



Note on Pontryagin maximum principle with running state constraints and smooth dynamics - Proof based on the Ekeland variational principle

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Note on Pontryagin maximum principle with running state constraints and smooth dynamics - Proof based on the Ekeland variational principle

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Abstract

In this note our aim is to give a proof of the Pontryagin maximum principle for a general optimal control problem with running state constraints and smooth dynamics. Our proof is based on the classical Ekeland variational principle.

The main result (and its proof) of this note are not new and are already well-known in the literature. The aim of the author is only to provide a complete and detailed proof of this classical theorem in the case of smooth dynamics. If you have any remarks or questions, do not hesitate to contact the author at *loic.bourdin@unilim.fr*.

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1 Main result

We first introduce some notations available throughout the paper. Let $T > 0$ be fixed. For every $n \in \mathbb{N}^*$ and every $1 \leq r \leq +\infty$, we denote by

- $\text{BF}_n := \text{BF}([0, T], \mathbb{R}^n)$ the classical space of bounded functions endowed with the classical uniform norm $\|\cdot\|_\infty$;
- $\text{C}_n := \text{C}([0, T], \mathbb{R}^n)$ the classical space of continuous functions endowed with $\|\cdot\|_\infty$;
- $\text{AC}_n := \text{AC}([0, T], \mathbb{R}^n)$ the classical space of absolutely continuous functions;
- $\text{BV}_n := \text{BV}([0, T], \mathbb{R}^n)$ the classical space of functions with bounded variations endowed with $\|\cdot\|_{\text{BV}_n}$ (see Appendix C for some recalls);
- $\text{L}_n^r := \text{L}^r([0, T], \mathbb{R}^n)$ the classical Lebesgue space of r -integrable functions endowed with its usual norm $\|\cdot\|_{\text{L}_n^r}$.

In the whole paper, when no confusion is possible, we remove the subscript n and we just denote by BF , C , AC , BV or L^r .

We denote by $\text{BF}_1^+ := \text{BF}([0, T], \mathbb{R}^+)$ and $\text{C}_1^+ := \text{C}([0, T], \mathbb{R}^+)$ where $\mathbb{R}^+ = [0, +\infty)$.

Then, $\eta \in \text{BV}_n$ is said to be *normalized* if $\eta(0) = 0$ and η is left-continuous on $(0, T)$. The subspace of normalized functions with bounded variations will be denoted by NBV_n .

Finally, the classical Lebesgue measure on $[0, T]$ will be denoted by λ .

1.1 A state constrained optimal control problem

Let m , n and $j \in \mathbb{N}^*$ be fixed. In this paper we consider the optimal control problem (**OCP**) given by

$$\begin{aligned}
& \text{minimize} && \Psi(q(T)), \\
& \text{subject to} && q \in \text{AC}_n, \ u \in \text{L}_m^\infty, \\
& && \dot{q}(t) = f(q(t), u(t), t), \quad \text{a.e. } t \in [0, T], \\
& && q(0) = q_0, \\
& && u(t) \in \Omega, \quad \text{a.e. } t \in [0, T], \\
& && G_i(q(t), t) \leq 0, \quad \forall t \in [0, T], \quad \forall i = 1, \dots, j,
\end{aligned} \tag{OCP}$$

where $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is of class C^1 , where $f : \mathbb{R}^n \times \mathbb{R}^m \times [0, T] \rightarrow \mathbb{R}^n$ is continuous and of class C^1 in its two first variables, where $G = (G_i)_{i=1, \dots, j} : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^j$ is continuous and of class C^1 in its first variable, and where $q_0 \in \mathbb{R}^n$ is fixed and $\Omega \subset \mathbb{R}^m$ is a nonempty closed subset.

Since Ψ , f and G are all regular, Problem (OCP) is said to be an optimal control problem *with smooth dynamics*. The last constraint corresponds to *running state constraints*.

We now introduce the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ associated to Problem (OCP) defined by

$$H(q, u, p, t) := \langle p, f(q, u, t) \rangle_{\mathbb{R}^n \times \mathbb{R}^n}.$$

1.2 Pontryagin maximum principle

Our main result in this note is given by the following theorem.

Theorem 1 (Pontryagin maximum principle). *Let $(q^*, u^*) \in \text{AC}_n \times \text{L}_m^\infty$ be an optimal solution of Problem (OCP). There exists a nontrivial couple (ψ, η) where $\psi \geq 0$ and $\eta = (\eta_i)_{i=1, \dots, j} \in \text{NBV}_j$ such that*

$$u^*(t) \in \arg \min_{v \in \Omega} H(q^*(t), v, p(t), t)$$

for a.e. $t \in [0, T]$, where $p \in \text{BV}_n$ is the unique global solution of the backward linear Cauchy-Stieltjes problem given by

$$\begin{cases} -dp = \partial_1 H(q^*, u^*, p, \cdot) dt + \sum_{i=1}^j \partial_1 G_i(q^*, \cdot) d\eta_i, & \text{on } [0, T], \\ p(T) = \psi \nabla \Psi(q^*(T)). \end{cases}$$

In addition, it holds that

$$\eta_i \text{ is monotonically increasing on } [0, T] \quad \text{and} \quad \int_0^T G_i(q^*(\tau), \tau) d\eta_i(\tau) = 0,$$

for every $i = 1, \dots, n$.

We refer to Appendix C for some recalls about functions of bounded variations and to Appendix D for details on linear Cauchy-Stieltjes problems.

2 Proof

This section is entirely devoted to the proof of Theorem 1. This proof is based on the classical Ekeland variational principle and is inspired from several references like [3, 11].

2.1 Preliminaries

Let $u \in \text{L}^\infty$. In this preliminary section we focus on the forward (nonlinear) Cauchy problem (CP_u) given by

$$\begin{cases} \dot{q}(t) = f(q(t), u(t), t), & \text{a.e. } t \in [0, T], \\ q(0) = q_0. \end{cases} \quad (\text{CP}_u)$$

A couple (q, I) is said to be a (local) solution of (CP_u) if

1. $I \subset [0, T]$ is an interval with nonempty interior such that $\min I = 0$;
2. $q : I \rightarrow \mathbb{R}^n$ is absolutely continuous on I and q satisfies

$$\begin{cases} \dot{q}(t) = f(q(t), u(t), t), & \text{a.e. } t \in I, \\ q(0) = q_0, \end{cases}$$

or equivalently, $q : I \rightarrow \mathbb{R}^n$ is continuous on I and q satisfies

$$q(t) = q_0 + \int_0^t f(q(\tau), u(\tau), \tau) d\tau,$$

for every $t \in I$.

The couple (q, I) is said to be a *global solution* of (\mathbf{CP}_u) if $I = [0, T]$.

Let (q, I) and (q', I') be two local solutions of (\mathbf{CP}_u) . We say that (q, I) is an *extension* of (q', I') if $I' \subset I$ and $q(t) = q'(t)$ for every $t \in I'$. We say that (q, I) is a *maximal solution* of (\mathbf{CP}_u) if it extends all other local solutions of (\mathbf{CP}_u) .

2.1.1 Some Cauchy-Lipschitz and continuous dependence results

Recall the two following classical Cauchy-Lipschitz (or Picard-Lindelöf) results.

Lemma 1. *For every $u \in L^\infty$, there exists a unique maximal solution of (\mathbf{CP}_u) .*

In the sequel we denote by $(q(\cdot, u), I(u))$ the maximal solution of (\mathbf{CP}_u) associated to $u \in L^\infty$.

Lemma 2. *Let $u \in L^\infty$. If $(q(\cdot, u), I(u))$ is not global (that is, $T \notin I(u)$) then $I(u)$ is not closed and $q(\cdot, u)$ is unbounded on $I(u)$.*

In the sequel we denote by $\mathcal{U} \subset L^\infty$ the set of controls $u \in L^\infty$ such that $T \in I(u)$. A control $u \in \mathcal{U}$ is usually said to be *admissible*. For every $u \in \mathcal{U}$ and every $R > \|u\|_{L^\infty}$, we introduce

$$K_{u,R} := \{(x, v, t) \in \mathbb{R}^n \times \mathbb{R}^m \times [0, T] \mid \|x - q(t, u)\|_{\mathbb{R}^n} \leq 1 \text{ and } \|v\|_{\mathbb{R}^m} \leq R\}.$$

From continuity of $q(\cdot, u)$ on $[0, T]$, $K_{u,R}$ is a compact subset of $\mathbb{R}^n \times \mathbb{R}^m \times [0, T]$. As a consequence, f , $\partial_1 f$ and $\partial_2 f$ are bounded on $K_{u,R}$ by some $L_{u,R} \geq 0$ and it holds that

$$\|f(x_2, v_2, t) - f(x_1, v_1, t)\|_{\mathbb{R}^n} \leq L_{u,R}(\|x_2 - x_1\|_{\mathbb{R}^n} + \|v_2 - v_1\|_{\mathbb{R}^m}), \quad (1)$$

for all $(x_1, v_1, t), (x_2, v_2, t) \in K_{u,R}$.

Proposition 1. *Let $u \in \mathcal{U}$. For every $R > \|u\|_{L^\infty}$, there exists $\nu_{u,R} > 0$ such that*

$$E_{u,R} := \overline{B}_{L^\infty}(0, R) \cap \overline{B}_{L^1}(u, \nu_{u,R})$$

is contained in \mathcal{U} . Moreover, for every $u' \in E_{u,R}$, it holds that $(q(\tau, u'), u'(\tau), \tau) \in K_{u,R}$ for a.e. $\tau \in [0, T]$.

Proof. Let $R > \|u\|_{L^\infty}$ and let $\nu_{u,R} > 0$ be such that $\nu_{u,R} L_{u,R} e^{TL_{u,R}} < 1$. Let $u' \in E_{u,R}$. Our aim is to prove that $T \in I(u')$. By contradiction, let us assume that the set

$$A := \{t \in I(u') \mid \|q(t, u') - q(t, u)\|_{\mathbb{R}^n} > 1\}$$

is not empty and let $t_0 := \inf A$. From continuity, it holds that $\|q(t_0, u') - q(t_0, u)\|_{\mathbb{R}^n} \geq 1$. Moreover, one has $t_0 > 0$ since $q(0, u') = q(0, u) = q_0$. Hence, $\|q(\tau, u') - q(\tau, u)\|_{\mathbb{R}^n} \leq 1$ for every $\tau \in [0, t_0]$. Therefore $(q(\tau, u'), u'(\tau), \tau)$ and $(q(\tau, u), u(\tau), \tau)$ belong to $K_{u,R}$ for a.e. $\tau \in [0, t_0]$. Since one has

$$q(t, u') - q(t, u) = \int_0^t f(q(\tau, u'), u'(\tau), \tau) - f(q(\tau, u), u(\tau), \tau) d\tau,$$

for every $t \in I(u')$, it follows from (1) that

$$\|q(t, u') - q(t, u)\|_{\mathbb{R}^n} \leq L_{u,R} \int_0^t \|u'(\tau) - u(\tau)\|_{\mathbb{R}^m} d\tau + L_{u,R} \int_0^t \|q(\tau, u') - q(\tau, u)\|_{\mathbb{R}^n} d\tau,$$

for every $t \in [0, t_0]$, which implies from the classical Gronwall lemma that

$$\|q(t, u') - q(t, u)\|_{\mathbb{R}^n} \leq L_{u,R} e^{TL_{u,R}} \|u' - u\|_{L^1} \leq \nu_R L_{u,R} e^{TL_{u,R}} < 1,$$

for every $t \in [0, t_0]$. This raises a contradiction at $t = t_0$. Therefore A is empty. We conclude that $q(\cdot, u')$ is bounded on $I(u')$, then $T \in I(u')$. Moreover, since A is empty, we also conclude that $\|q(t, u') - q(t, u)\|_{\mathbb{R}^n} \leq 1$ for every $t \in [0, T]$, and thus $(q(\tau, u'), u'(\tau), \tau) \in K_{u,R}$ for a.e. $\tau \in [0, T]$. \square

We conclude this section with the following continuous dependence result.

Proposition 2. *Let $u \in \mathcal{U}$ and $R > \|u\|_{L^\infty}$. The mapping*

$$\begin{aligned} F_{u,R} : (E_{u,R}, \|\cdot\|_{L^1}) &\longrightarrow (C_n, \|\cdot\|_\infty) \\ u' &\longmapsto q(\cdot, u') \end{aligned}$$

is $C_{u,R}$ -Lipschitz continuous for some $C_{u,R} \geq 0$.

Proof. Let u' and u'' be two elements of $E_{u,R} \subset \mathcal{U}$. We know that $(q(\tau, u''), u''(\tau), \tau)$ and $(q(\tau, u'), u'(\tau), \tau)$ are elements of $K_{u,R}$ for a.e. $\tau \in [0, T]$. Following the same arguments as in the previous proof, it follows that

$$\|q(t, u'') - q(t, u')\|_{\mathbb{R}^n} \leq L_{u,R} e^{TL_{u,R}} \|u'' - u'\|_{L^1},$$

for every $t \in [0, T]$. The lemma follows with $C_{u,R} := L_{u,R} e^{TL_{u,R}} \geq 0$. \square

2.1.2 Implicit spike variations and a differentiable dependence result

Before introducing the concept of implicit spike variations, we first need to recall the following lemma (see [11, Paragraph 3.2 p.143]). The proof is recalled in Appendix A.

Lemma 3. *Let $h \in L_n^1$. Then, for all $\rho \in (0, 1)$, there exists a measurable set $Q_\rho \subset [0, T]$ such that $\lambda(Q_\rho) = \rho T$ and*

$$\sup_{t \in [0, T]} \left\| \int_0^t \left(1 - \frac{1}{\rho} \mathbf{1}_{Q_\rho}(s) \right) h(s) ds \right\|_{\mathbb{R}^n} \leq \rho.$$

Note that Q_ρ depends on h .

Let $u \in \mathcal{U}$ and $u' \in L^\infty$. For every $\rho \in [0, 1)$, we introduce the so-called *implicit spike variation* $u(\cdot, \rho)$ of u associated to u' as

$$u(\tau, \rho) := \begin{cases} u'(\tau) & \text{if } \tau \in Q_\rho, \\ u(\tau) & \text{if } \tau \notin Q_\rho, \end{cases}$$

for a.e. $\tau \in [0, T]$, where Q_ρ is defined in Lemma 3 associated to $h_{u,u'} \in L_n^\infty \subset L_n^1$ defined by

$$h_{u,u'}(\tau) := f(q(\tau, u), u'(\tau), \tau) - f(q(\tau, u), u(\tau), \tau),$$

for a.e. $\tau \in [0, T]$.¹

¹For $\rho = 0$, we fix $Q_\rho = \emptyset$.

Finally, we introduce the so-called *variation vector* $w(\cdot, u, u')$ associated to (u, u') as the unique maximal solution, which is moreover global (see Appendix D.2), of the forward linear Cauchy problem given by

$$\begin{cases} \dot{w}(t) = \partial_1 f(q(t, u), u(t), t) \times w(t) + h_{u, u'}(t), & \text{a.e. } t \in [0, T], \\ w(0) = 0. \end{cases}$$

Let us prove the following differentiability dependence result.

Proposition 3. *The mapping $F_{u, u'}$ defined by*

$$F_{u, u'}(\rho) := q(\cdot, u(\cdot, \rho)) \in C_n,$$

for sufficiently small $\rho \geq 0$, is Fréchet-differentiable at $\rho = 0$, with $DF_{u, u'}(0) = w(\cdot, u, u')$.

Proof. Let $R := \max(\|u\|_{L^\infty} + 1, \|u'\|_{L^\infty})$. Since $\lambda(Q_\rho) = \rho T$ (see Lemma 3), it holds that $\|u(\cdot, \rho) - u\|_{L^1} \leq 2RT\rho$ for every $\rho \in [0, 1]$. As a consequence, for sufficiently small $\rho \geq 0$, $u(\cdot, \rho) \in E_{u, R} \subset \mathcal{U}$ and then $F_{u, u'}(\rho)$ is well-defined. Moreover, it follows from Proposition 2 that $\|q(\cdot, u(\cdot, \rho)) - q(\cdot, u)\|_\infty \leq 2RTC_{u, R}\rho$, and consequently $q(\cdot, u(\cdot, \rho))$ uniformly converges on $[0, T]$ to $q(\cdot, u)$.

Let us assume by contradiction that $F_{u, u'}$ is not Fréchet-differentiable at $\rho = 0$ with $DF_{u, u'}(0) = w(\cdot, u, u')$. Then, there exists $\varepsilon > 0$ et $(\rho_k)_k$ a positive sequence such that $(\rho_k)_k$ tends to zero and such that

$$\left\| \frac{F_{u, u'}(\rho_k) - F_{u, u'}(0)}{\rho_k} - w(\cdot, u, u') \right\|_\infty \geq \varepsilon$$

for all $k \in \mathbb{N}$. In this proof, for the ease of notations, we denote by $w := w(\cdot, u, u')$, $q := q(\cdot, u)$ and by $q_k := q(\cdot, u(\cdot, \rho_k))$, $u_k := u(\cdot, \rho_k)$ for every $k \in \mathbb{N}$. Since the sequence $(u_k)_k$ converges to u in L^1 , we deduce from the (partial) converse of the classical Lebesgue dominated convergence theorem that there exists a subsequence (that we do not relabel) such that $(u_k)_k$ tends to u a.e. on $[0, T]$. For every $k \in \mathbb{N}$ and every $t \in [0, T]$, we define $z_k(t) := \frac{q_k(t) - q(t)}{\rho_k} - w(t)$. From our assumption, it holds that $\|z_k\|_\infty \geq \varepsilon$ for all $k \in \mathbb{N}$. On the other hand, we have

$$z_k(t) = \int_0^t \frac{f(q_k(\tau), u_k(\tau), \tau) - f(q(\tau), u(\tau), \tau)}{\rho_k} - \partial_1 f(q(\tau), u(\tau), \tau) \times w(\tau) - h_{u, u'}(\tau) d\tau,$$

that is,

$$\begin{aligned} z_k(t) = \int_0^t & \frac{f(q_k(\tau), u_k(\tau), \tau) - f(q(\tau), u_k(\tau), \tau)}{\rho_k} - \partial_1 f(q(\tau), u(\tau), \tau) \times w(\tau) \\ & + \frac{f(q(\tau), u_k(\tau), \tau) - f(q(\tau), u(\tau), \tau)}{\rho_k} - h_{u, u'}(\tau) d\tau, \end{aligned}$$

for every $t \in [0, T]$. From the classical Taylor formula with integral rest, we obtain that

$$\begin{aligned} z_k(t) = \int_0^t & \partial_1 f(q(\tau), u(\tau), \tau) \times z_k(\tau) d\tau + \int_0^t \left(\frac{1}{\rho_k} \mathbf{1}_{Q_{\rho_k}}(\tau) - 1 \right) h_{u, u'}(\tau) d\tau \\ & + \int_0^t \left[\int_0^1 \partial_1 f(q(\tau) + \theta(q_k(\tau) - q(\tau)), u_k(\tau), \tau) d\theta - \partial_1 f(q(\tau), u(\tau), \tau) \right] \frac{q_k(\tau) - q(\tau)}{\rho_k} d\tau, \end{aligned}$$

for every $t \in [0, T]$. Hence, from Lemma 3, we get that

$$\|z_k(t)\|_{\mathbb{R}^n} \leq \rho_k + 2RTC_{u,R}\kappa_k + L_{u,R} \int_0^t \|z_k(\tau)\|_{\mathbb{R}^n} d\tau,$$

for every $t \in [0, T]$, where

$$\kappa_k := \int_0^T \int_0^1 \|\partial_1 f(q(\tau) + \theta(q_k(\tau) - q(\tau)), u_k(\tau), \tau) - \partial_1 f(q(\tau), u(\tau), \tau)\|_{\mathbb{R}^n \times \mathbb{R}^n} d\theta d\tau.$$

From the continuity and the boundedness of $\partial_1 f$ on $K_{u,R}$, since $(u_k)_k$ tends to u a.e. on $[0, T]$ and from the classical Lebesgue dominated convergence theorem, one can easily prove that $(\kappa_k)_k$ tends to zero. Finally, from the classical Gronwall lemma, we obtain that $\|z_k(t)\|_{\mathbb{R}^n} \leq (\rho_k + 2RTC_{u,R}\kappa_k)e^{TL_{u,R}}$ for every $t \in [0, T]$. This raises a contradiction with the inequality $\|z_k\|_{\infty} \geq \varepsilon$ for all $k \in \mathbb{N}$. The proof is complete. \square

2.2 Application of the Ekeland variational principle

Let us introduce $g = (g_i)_{i=1,\dots,j} : C_n \rightarrow C_j$ the application defined by $g(q) := G(q, \cdot)$ for every $q \in C_n$, and let S be the nonempty closed convex cone of C_j defined by $S := C([0, T], (\mathbb{R}^-)^j)$. Thus the running state constraints in Problem (OCP) can equivalently be replaced by

$$g(q) \in S.$$

Note that g is of class C^1 with $Dg(q)(w) = \partial_1 G(q, \cdot) \times w$ for every $q, w \in C_n$, and that S has a nonempty interior.

Since $(C_j, \|\cdot\|_{\infty})$ is a separable Banach space, we endow C_j with an equivalent norm $\|\cdot\|_{C_j}$ such that the associated dual norm $\|\cdot\|_{C_j^*}$ is strictly convex (see Proposition 4 in Appendix B.1). Then, we denote by d_S the 1-Lipschitz continuous distance function to S defined by $d_S(q) := \inf_{z \in S} \|q - z\|_{C_j}$ for every $q \in C_j$. Since the dual norm $\|\cdot\|_{C_j^*}$ is strictly convex, we know that d_S is strictly Hadamard-differentiable on $C_j \setminus S$ with $\|Dd_S(q)\|_{C_j^*} = 1$ for every $q \in C_j \setminus S$ (see Proposition 5 in Appendix B.2). As a consequence, d_S^2 is also strictly Hadamard-differentiable on $C_j \setminus S$ with $Dd_S^2(q) = 2d_S(q)Dd_S(q)$ for every $q \in C_j \setminus S$. We also recall that d_S^2 is Fréchet-differentiable on S with $Dd_S^2(q) = 0$ for every $q \in S$ (see Remark 6 in Appendix B.2).

In the whole section, let $q(\cdot, u^*) \in AC$ and $u^* \in L^\infty$ be an optimal solution of Problem (OCP). Let $(R_\ell)_\ell$ be a positive sequence such that $R_\ell > \|u^*\|_{L^\infty}$ for every $\ell \in \mathbb{N}$ and such that $\lim_\ell R_\ell = +\infty$. Let $(\varepsilon_k)_k$ be a positive sequence such that $\lim_k \varepsilon_k = 0$. For every $\ell, k \in \mathbb{N}$, we consider the penalized functional given by

$$\begin{aligned} J_k^\ell : E_{u^*, R_\ell}^\Omega &\longrightarrow \mathbb{R}_*^+ \\ u &\longmapsto \sqrt{\left((\Psi(q(T, u)) - \Psi(q(T, u^*)) + \varepsilon_k)^+\right)^2 + d_S^2(g(q(\cdot, u)))}, \end{aligned}$$

where

$$E_{u^*, R_\ell}^\Omega := \{u \in E_{u^*, R_\ell} \mid u(\tau) \in \Omega \text{ for a.e. } \tau \in [0, T]\}.$$

Note that J_k^ℓ is a positive functional because of the optimality of u^* . We endow E_{u^*, R_ℓ}^Ω with the classical norm $\|\cdot\|_{L^1}$. Since Ω is a nonempty closed subset of \mathbb{R}^m , it follows from the (partial) converse of the classical Lebesgue dominated convergence theorem that $(E_{u^*, R_\ell}^\Omega, \|\cdot\|_{L^1})$ is a nonempty

closed subset of $(L^1, \|\cdot\|_{L^1})$ and consequently $(E_{u^*, R_\ell}^\Omega, \|\cdot\|_{L^1})$ is a complete metric space. Moreover, from the continuities of Ψ , F_{u^*, R_ℓ} (see Proposition 2), d_S and of g , one can easily see that J_k^ℓ is continuous on $(E_{u^*, R_\ell}^\Omega, \|\cdot\|_{L^1})$.

Moreover it holds that $J_k^\ell(u^*) = \varepsilon_k$. As a consequence, from the classical Ekeland variational principle, we conclude that for every $\ell, k \in \mathbb{N}$, there exists $u_k^\ell \in E_{u^*, R_\ell}^\Omega$ such that $\|u_k^\ell - u^*\|_{L^1} \leq \sqrt{\varepsilon_k}$ and

$$-\sqrt{\varepsilon_k} \|u - u_k^\ell\|_{L^1} \leq J_k^\ell(u) - J_k^\ell(u_k^\ell), \quad (2)$$

for all $u \in E_{u^*, R_\ell}^\Omega$. In particular, for a fixed $\ell \in \mathbb{N}$, note that the sequence $(u_k^\ell)_k$ converges to u^* in L^1 and consequently, the sequence $(q(\cdot, u_k^\ell))_k$ uniformly converges on $[0, T]$ to $q(\cdot, u^*)$ (see Proposition 2).

For every $\ell, k \in \mathbb{N}$, we introduce

$$\psi_k^\ell := \frac{1}{J_k^\ell(u_k^\ell)} (\Psi(q(T, u_k^\ell)) - \Psi(q(T, u^*)) + \varepsilon_k)^+ \geq 0,$$

and

$$\varphi_k^\ell := \begin{cases} \frac{1}{J_k^\ell(u_k^\ell)} d_S(g(q(\cdot, u_k^\ell))) Dd_S(g(q(\cdot, u_k^\ell))) \in C_j^* & \text{if } g(q(\cdot, u_k^\ell)) \notin S, \\ 0 \in C_j^* & \text{if } g(q(\cdot, u_k^\ell)) \in S. \end{cases}$$

In particular it holds that $|\psi_k^\ell|^2 + \|\varphi_k^\ell\|_{C_j^*}^2 = 1$ for every $\ell, k \in \mathbb{N}$.

In the sequel our aim is to derive some important inequalities from Inequality (2) using implicit spike variations on u_k^ℓ .

Remark 1. In this remark (and in Remarks 2 and 3), our aim is to provide two crucial inequalities satisfied by φ_k^ℓ . In the case $g(q(\cdot, u_k^\ell)) \notin S$, recall that $Dd_S(g(q(\cdot, u_k^\ell)))$ belongs to the subdifferential of d_S at the point $g(q(\cdot, u_k^\ell))$. As a consequence, in both cases $g(q(\cdot, u_k^\ell)) \notin S$ and $g(q(\cdot, u_k^\ell)) \in S$, it holds that

$$\langle \varphi_k^\ell, z - g(q(\cdot, u_k^\ell)) \rangle_{C_j^* \times C_j} \leq 0, \quad (3)$$

for every $z \in S$. Since S has a nonempty interior, there exists $\xi \in S$ and $\delta > 0$ such that $\xi + \delta z \in S$ for every $z \in \overline{B}_{(C_j, \|\cdot\|_{C_j})}(0, 1)$. As a consequence, we obtain that

$$\delta \langle \varphi_k^\ell, z \rangle_{C_j^* \times C_j} \leq \langle \varphi_k^\ell, g(q(\cdot, u_k^\ell)) - \xi \rangle_{C_j^* \times C_j},$$

for every $z \in \overline{B}_{(C_j, \|\cdot\|_{C_j})}(0, 1)$. We deduce that

$$\delta \|\varphi_k^\ell\|_{C_j^*} = \delta \sqrt{1 - |\psi_k^\ell|^2} \leq \langle \varphi_k^\ell, g(q(\cdot, u_k^\ell)) - \xi \rangle_{C_j^* \times C_j}. \quad (4)$$

2.2.1 First inequality depending on ℓ fixed

In this section, we fix $\ell \in \mathbb{N}$. Recall that the sequence $(u_k^\ell)_k$ converges to u^* in L^1 . Using compactness arguments, we infer the existence of a subsequence of $(\varepsilon_k)_k$ (that we do not relabel)² such that $(u_k^\ell)_k$ converges to u^* a.e. on $[0, T]$, $(\psi_k^\ell)_k$ converges to some $\psi^\ell \geq 0$ and $(\varphi_k^\ell)_k$ weakly* converges

²The subsequence of $(\varepsilon_k)_k$ is not relabel. However, it is worth to note that the extracted subsequence depends on ℓ fixed.

to some $\varphi^\ell \in C_j^*$. In particular, it holds that $|\psi^\ell|^2 + \|\varphi^\ell\|_{C_j^*}^2 \leq 1$.

In the whole section, for the ease of notations, we denote by $q_k^\ell := q(\cdot, u_k^\ell)$ for every $k \in \mathbb{N}$. Let $u' \in L^\infty$ such that $u'(\tau) \in \Omega \cap \overline{B}_{\mathbb{R}^m}(0, R_\ell)$ for a.e. $\tau \in [0, T]$. For every $\rho \in [0, 1]$, we consider the implicit spike variation

$$u_k^\ell(\tau, \rho) := \begin{cases} u'(\tau) & \text{if } \tau \in Q_\rho, \\ u_k^\ell(\tau) & \text{if } \tau \notin Q_\rho, \end{cases}$$

for a.e. $\tau \in [0, T]$, where Q_ρ is defined in Lemma 3 associated to $h_{u_k^\ell, u'} \in L_n^\infty \subset L_n^1$ defined by

$$h_{u_k^\ell, u'}(\tau) := f(q_k^\ell(\tau), u'(\tau), \tau) - f(q_k^\ell(\tau), u_k^\ell(\tau), \tau),$$

for a.e. $\tau \in [0, T]$.³

First of all, note that $\|u_k^\ell(\cdot, \rho) - u^*\|_{L^1} \leq \|u_k^\ell(\cdot, \rho) - u_k^\ell\|_{L^1} + \|u_k^\ell - u^*\|_{L^1} \leq 2R_\ell T\rho + \sqrt{\varepsilon_k} < \nu_{u^*, R_\ell}$ for sufficiently small ρ and sufficiently large k and then $u_k^\ell(\cdot, \rho) \in E_{u^*, R_\ell}^\Omega$. For such a sufficiently small ρ and sufficiently large k , we apply Inequality (2) with $u = u_k^\ell(\cdot, \rho)$ and we obtain that

$$-2R_\ell T\sqrt{\varepsilon_k} \leq \frac{J_k^\ell(u_k^\ell(\cdot, \rho)) - J_k^\ell(u_k^\ell)}{\rho} = \frac{J_k^\ell(u_k^\ell(\cdot, \rho))^2 - J_k^\ell(u_k^\ell)^2}{\rho} \times \frac{1}{J_k^\ell(u_k^\ell(\cdot, \rho)) + J_k^\ell(u_k^\ell)}.$$

From the continuity of J_k^ℓ , we get that $\lim_{\rho \rightarrow 0} J_k^\ell(u_k^\ell(\cdot, \rho)) + J_k^\ell(u_k^\ell) = 2J_k^\ell(u_k^\ell)$. From the differentiability of the application $x \mapsto (x^+)^2$ for $x \in \mathbb{R}$, of Ψ , of g , of d_S^2 and of $F_{u_k^\ell, u'}$ (see Proposition 3), we obtain that

$$\begin{aligned} \lim_{\rho \rightarrow 0} \frac{J_k^\ell(u_k^\ell(\cdot, \rho))^2 - J_k^\ell(u_k^\ell)^2}{\rho} &= 2 \left(\Psi(q_k^\ell(T)) - \Psi(q(T, u^*)) + \varepsilon_k \right)^+ \left\langle \nabla \Psi(q_k^\ell(T)), w(T, u_k^\ell, u') \right\rangle_{\mathbb{R}^n \times \mathbb{R}^n} \\ &\quad + \left\langle 2d_S(g(q_k^\ell))Dd_S(g(q_k^\ell)), Dg(q_k^\ell)(w(\cdot, u_k^\ell, u')) \right\rangle_{C_j^* \times C_j}, \end{aligned}$$

with the convention that the second term is zero if $g(q_k^\ell) \in S$. Finally, we have obtained that

$$-2R_\ell T\sqrt{\varepsilon_k} \leq \psi_k^\ell \left\langle \nabla \Psi(q_k^\ell(T)), w(T, u_k^\ell, u') \right\rangle_{\mathbb{R}^n \times \mathbb{R}^n} + \left\langle \varphi_k^\ell, Dg(q_k^\ell)(w(\cdot, u_k^\ell, u')) \right\rangle_{C_j^* \times C_j}. \quad (5)$$

To conclude this section, we need the following result.⁴

Lemma 4. *The sequence $(w(\cdot, u_k^\ell, u'))_k$ uniformly converges on $[0, T]$ to $w(\cdot, u^*, u')$.*

Proof. In this proof, for the ease of notations, we denote by $q^* := q(\cdot, u^*)$, $w := w(\cdot, u^*, u')$ and by $w_k := w(\cdot, u_k^\ell, u')$ for all $k \in \mathbb{N}$. It holds that

$$\begin{aligned} w_k(t) - w(t) &= \int_0^t \partial_1 f(q_k^\ell(\tau), u_k^\ell(\tau), \tau) \times w_k(\tau) + h_{u_k^\ell, u'}(\tau) \\ &\quad - \partial_1 f(q^*(\tau), u^*(\tau), \tau) \times w(\tau) - h_{u^*, u'}(\tau) d\tau, \end{aligned}$$

³For $\rho = 0$, we fix $Q_\rho = \emptyset$.

⁴This result requires to fix $\ell \in \mathbb{N}$. Indeed, one needs a bound on $\|u_k^\ell\|_{L^\infty}$ in order to conclude from the classical Lebesgue dominated convergence theorem.

that is,

$$\begin{aligned} w_k(t) - w(t) &= \int_0^t \partial_1 f(q_k^\ell(\tau), u_k^\ell(\tau), \tau) \times (w_k(\tau) - w(\tau)) d\tau + \int_0^t h_{u_k^\ell, u'}(\tau) - h_{u^*, u'}(\tau) d\tau \\ &\quad + \int_0^t \left(\partial_1 f(q_k^\ell(\tau), u_k^\ell(\tau), \tau) - \partial_1 f(q^*(\tau), u^*(\tau), \tau) \right) \times w(\tau) d\tau, \end{aligned}$$

for every $t \in [0, T]$. Recall that $u_k^\ell \in E_{u^*, R_\ell}$ and $u' \in \overline{B}_{L^\infty}(0, R_\ell)$, then $(q_k^\ell(\tau), u_k^\ell(\tau), \tau) \in K_{u^*, R_\ell}$ (see Proposition 1) and $(q_k^\ell(\tau), u'(\tau), \tau) \in K_{u^*, R_\ell}$ for a.e. $\tau \in [0, T]$ and recall that f and $\partial_1 f$ are bounded on K_{u^*, R_ℓ} by $L_{u^*, R_\ell} \geq 0$. Recall also that $(u_k^\ell)_k$ tends to u^* a.e. on $[0, T]$. Finally, using similar arguments than in the proof of Proposition 3 and the classical Gronwall lemma, one can easily conclude the proof. \square

Using the above lemma and the C^1 -regularity of Ψ and g , by letting k tend to $+\infty$ in Inequality (5), we obtain that

$$0 \leq \psi^\ell \left\langle \nabla \Psi(q(T, u^*)), w(T, u^*, u') \right\rangle_{\mathbb{R}^n \times \mathbb{R}^n} + \left\langle \varphi^\ell, Dg(q(\cdot, u^*))(w(\cdot, u^*, u')) \right\rangle_{C_j^* \times C_j}. \quad (6)$$

Remark 2. Letting k tend to $+\infty$ in Remark 1, one can easily obtain the two following crucial inequalities:

$$\langle \varphi^\ell, z - g(q(\cdot, u^*)) \rangle_{C_j^* \times C_j} \leq 0, \quad (7)$$

for every $z \in S$, and

$$\delta \sqrt{1 - |\psi^\ell|^2} \leq \langle \varphi^\ell, g(q(\cdot, u^*)) - \xi \rangle_{C_j^* \times C_j}. \quad (8)$$

2.2.2 Second inequality independent of ℓ

In the previous section, we have obtained Inequality (6) that is valid for a fixed $\ell \in \mathbb{N}$ and for every $u' \in L^\infty$ such that $u'(\tau) \in \Omega \cap \overline{B}_{\mathbb{R}^m}(0, R_\ell)$. Our aim in this section is to remove the dependence in R_ℓ (in order to cover the case where Ω is unbounded).

Since $|\psi^\ell|^2 + \|\varphi^\ell\|_{C_j^*}^2 \leq 1$ for every $\ell \in \mathbb{N}$ and from compactness arguments, we infer the existence of a subsequence of $(R_\ell)_\ell$ (that we do not relabel) such that $(\psi^\ell)_\ell$ converges to some $\psi \geq 0$ and $(\varphi^\ell)_\ell$ weakly* converges to some $\varphi \in C_j^*$.

Let $u' \in L^\infty$ such that $u'(\tau) \in \Omega$ for a.e. $\tau \in [0, T]$. Let $\ell \in \mathbb{N}$ be sufficiently large in order to have $R_\ell > \|u'\|_{L^\infty}$. From Inequality (6), it holds that

$$0 \leq \psi^\ell \left\langle \nabla \Psi(q(T, u^*)), w(T, u^*, u') \right\rangle_{\mathbb{R}^n \times \mathbb{R}^n} + \left\langle \varphi^\ell, Dg(q(\cdot, u^*))(w(\cdot, u^*, u')) \right\rangle_{C_j^* \times C_j}.$$

Letting ℓ tend to $+\infty$, we prove that

$$0 \leq \psi \left\langle \nabla \Psi(q(T, u^*)), w(T, u^*, u') \right\rangle_{\mathbb{R}^n \times \mathbb{R}^n} + \left\langle \varphi, Dg(q(\cdot, u^*))(w(\cdot, u^*, u')) \right\rangle_{C_j^* \times C_j}, \quad (9)$$

for every $u' \in L^\infty$ such that $u'(\tau) \in \Omega$ for a.e. $\tau \in [0, T]$.

Remark 3. Letting ℓ tend to $+\infty$ in Remark 2, one can easily obtain the two following crucial inequalities:

$$\langle \varphi, z - g(q(\cdot, u^*)) \rangle_{C_j^* \times C_j} \leq 0, \quad (10)$$

for every $z \in S$, and

$$\delta \sqrt{1 - |\psi|^2} \leq \langle \varphi, g(q(\cdot, u^*)) - \xi \rangle_{C_j^* \times C_j}. \quad (11)$$

Inequality (11) proves that the couple (ψ, φ) is not trivial.

2.3 Introduction of the adjoint vector p

2.3.1 Introduction of η

Let us denote by $\varphi = (\varphi_i)_{i=1, \dots, j}$ where $\varphi_i \in C_1^*$ and let us apply the classical Riesz theorem (see Proposition 6 in Appendix C.1). For every $i = 1, \dots, j$, there exists a unique $\eta_i \in \text{NBV}_1$ such that

$$\langle \varphi_i, z \rangle_{C_1^* \times C_1} = \int_0^T z(\tau) d\eta_i(\tau),$$

for every $z \in C_1$. Recall that $\varphi_i = 0$ if and only if $\eta_i = 0$. As a consequence, from Remark 3, the couple (ψ, η) is not trivial, where $\eta := (\eta_i)_{i=1, \dots, j} \in \text{NBV}_j$.

Taking

$$z = \left(g_1(q(\cdot, u^*)), \dots, g_{i-1}(q(\cdot, u^*)), 0, g_{i+1}(q(\cdot, u^*)), \dots, g_j(q(\cdot, u^*)) \right) \in S$$

and

$$z = \left(g_1(q(\cdot, u^*)), \dots, g_{i-1}(q(\cdot, u^*)), 2g_i(q(\cdot, u^*)), g_{i+1}(q(\cdot, u^*)), \dots, g_j(q(\cdot, u^*)) \right) \in S$$

in Inequality (10), we obtain that $\langle \varphi_i, g_i(q(\cdot, u^*)) \rangle_{C_1^* \times C_1} = 0$, that is,

$$\int_0^T G_i(q(\tau, u^*), \tau) d\eta_i(\tau) = 0,$$

for every $i = 1, \dots, n$.

Moreover, it follows that $\langle \varphi_i, z \rangle_{C_1^* \times C_1} \geq 0$ for every $z \in C_1^+$. From the classical Riesz theorem (see Proposition 6 in Appendix C.1), we deduce that η_i is monotonically increasing on $[0, T]$ for every $i = 1, \dots, j$.

2.3.2 Definition of p

Using notations introduced in Appendix C.2, one has

$$\langle \varphi, z \rangle_{C_j^* \times C_j} = \int_0^T \langle z(\tau), d\eta(\tau) \rangle$$

for every $z \in C_j$. From Inequality (9) and since $Dg(q)(w) = \partial_1 G(q, \cdot) \times w$ for every $q, w \in C_n$, we have proved that

$$0 \leq \psi \left\langle \nabla \Psi(q(T, u^*)), w(T, u^*, u') \right\rangle_{\mathbb{R}^n \times \mathbb{R}^n} + \int_0^T \left\langle \partial_1 G(q(\tau, u^*), \tau) \times w(\tau, u^*, u'), d\eta(\tau) \right\rangle, \quad (12)$$

for every $u' \in L^\infty$ such that $u'(\tau) \in \Omega$ for a.e. $\tau \in [0, T]$.

Let $Z(\cdot, \cdot)$ be the state-transition matrix associated to $\partial_1 f(q(\cdot, u^*), u^*, \cdot) \in L^\infty([0, T], \mathbb{R}^{n,n})$ (see Appendix D.1). From the classical Duhamel formula (see Proposition 8 in Appendix D.2), it holds that

$$w(t, u^*, u') = \int_0^t Z(t, s) \times h_{u^*, u'}(s) ds$$

for every $t \in [0, T]$. Replacing $w(\cdot, u^*, u')$ in (12), using first the Fubini-type formula (18) and then Equality (17), one can obtain that

$$0 \leq \int_0^T \left\langle h_{u^*, u'}(s), \psi Z(T, s)^\top \times \nabla \Psi(q(T, u^*)) + \int_s^T Z(\tau, s)^\top \times \partial_1 G(q(\tau, u^*), \tau)^\top \times d\eta(\tau) \right\rangle_{\mathbb{R}^n \times \mathbb{R}^n} ds, \quad (13)$$

for every $u' \in L^\infty$ such that $u'(\tau) \in \Omega$ for a.e. $\tau \in [0, T]$.

Let $p \in BV_n$ be the unique global solution of the backward linear Cauchy-Stieltjes problem given by

$$\begin{cases} -dp = \partial_1 f(q(\cdot, u^*), u^*, \cdot)^\top \times p dt + \sum_{i=1}^j \partial_1 G_i(q(\cdot, u^*), \cdot) d\eta_i, & \text{on } [0, T], \\ p(T) = \psi \nabla \Psi(q(T, u^*)). \end{cases}$$

We refer to Proposition 9 in Appendix D.3 for the existence and uniqueness of p . Note that p is independent of u' . From the Duhamel-type formula (see Proposition 10 in Appendix D.3), it holds that

$$p(s) = \psi Z(T, s)^\top \times \nabla \Psi(q(T, u^*)) + \int_s^T Z(\tau, s)^\top \times \partial_1 G(q(\tau, u^*), \tau)^\top \times d\eta(\tau) \in \mathbb{R}^n,$$

for every $s \in [0, T]$. It follows from the above expression of p and from Inequality (13) that $\int_0^T \langle h_{u^*, u'}(s), p(s) \rangle_{\mathbb{R}^n \times \mathbb{R}^n} ds \geq 0$, that is,

$$\int_0^T H(q(s, u^*), u'(s), p(s), s) - H(q(s, u^*), u^*(s), p(s), s) ds \geq 0, \quad (14)$$

for every $u' \in L^\infty$ such that $u'(\tau) \in \Omega$ for a.e. $\tau \in [0, T]$.

2.4 End of the proof

Let $v \in \Omega$ be fixed. Let $t \in [0, T]$ be a continuity point of $p \in BV_n$ and be a Lebesgue point of the application $s \mapsto H(q(s, u^*), u^*(s), p(s), s)$ which belongs to L_1^∞ . Let $\alpha \in (0, T - t)$ and let us consider

$$u'(\tau) := \begin{cases} v & \text{if } \tau \in [t, t + \alpha), \\ u^*(\tau) & \text{if } \tau \notin [t, t + \alpha), \end{cases}$$

for a.e. $\tau \in [0, T]$. From Inequality (14), it holds that

$$\int_t^{t+\alpha} H(q(s, u^*), v, p(s), s) - H(q(s, u^*), u^*(s), p(s), s) ds \geq 0.$$

Dividing by $\alpha > 0$ and letting $\alpha \rightarrow 0^+$, we obtain that

$$H(q(t, u^*), v, p(t), t) - H(q(t, u^*), u^*(t), p(t), t) \geq 0.$$

Since the last inequality is true for every $v \in \Omega$ and for a.e. $t \in [0, T]$, we obtain the maximization condition

$$u^*(t) \in \arg \min_{v \in \Omega} H(q(t, u^*), v, p(t), t)$$

for a.e. $t \in [0, T]$.

A Proof of Lemma 3

Recall that the classical Lebesgue measure λ is a nonatomic measure (see, e.g., [8, Remark 1.161 p.111]). As a consequence, from the classical Sierpinski (or Lyapunov) theorem (see [14] or [9, p.37]), for all measurable set $R \subset [0, T]$, there exists a measurable set $R_\rho \subset R$ such that $\lambda(R_\rho) = \rho\lambda(R)$ for all $\rho \in (0, 1)$.

The whole section is dedicated to the proof of Lemma 3. Let $\rho \in (0, 1)$.

Lemma 5. *Let $b : [0, T]^2 \rightarrow \mathbb{R}^n$ be defined by*

$$b(t, s) := h(s)\mathbf{1}_{[0, t]}(s).$$

Then, $b \in C([0, T], L_n^1)$.

Proof. Let $t \in [0, T]$ and let $(t_n) \subset [0, T]$ be a decreasing sequence such that $t_n \rightarrow t$. Then, it holds that

$$\begin{aligned} \|b(t_n, \cdot) - b(t, \cdot)\|_{L^1} &= \int_0^T |b(t_n, s) - b(t, s)| ds \\ &= \int_0^t |b(t_n, s) - b(t, s)| ds + \int_t^{t_n} |b(t_n, s) - b(t, s)| ds = \int_t^{t_n} |h(s)| ds \rightarrow 0. \end{aligned}$$

Similarly, we prove that $\|b(t_n, \cdot) - b(t, \cdot)\|_{L^1} \rightarrow 0$ for any increasing sequence $(t_n) \subset [0, T]$ such that $t_n \rightarrow t$. The proof is complete. \square

Since $[0, T]$ is compact, there exists $\delta > 0$ such that $\|b(t, \cdot) - b(\bar{t}, \cdot)\|_{L^1} \leq \frac{\rho^2}{2(\rho+1)}$ for all $t, \bar{t} \in [0, T]$ satisfying $|t - \bar{t}| < \delta$. In the sequel, we fix $0 = t_0 < t_1 < \dots < t_N = T$ such that $|t_{r+1} - t_r| < \delta$ for all $r = 0, \dots, N-1$ and we define

$$B(\cdot) := (b(t_0, \cdot), b(t_1, \cdot), \dots, b(t_N, \cdot)) \in L^1([0, T], (\mathbb{R}^n)^{N+1}).$$

Lemma 6. *There exists a measurable set $Q_\rho \subset [0, T]$ such that $\lambda(Q_\rho) = \rho T$ and*

$$\left\| \int_0^T \left(1 - \frac{1}{\rho} \mathbf{1}_{Q_\rho}(s)\right) B(s) ds \right\|_{(\mathbb{R}^n)^{N+1}} \leq \frac{\rho}{2}.$$

Proof. Since $B \in L^1([0, T], (\mathbb{R}^n)^{N+1})$, there exists a simple function $J : [0, T] \rightarrow (\mathbb{R}^n)^{N+1}$ such that $\int_0^T \|B(s) - J(s)\|_{(\mathbb{R}^n)^{N+1}} ds \leq \frac{\rho^2}{2(\rho+1)}$. Let us denote by $J := \sum_{i=1}^K a_i \mathbf{1}_{R^i}$, where $a_i \in (\mathbb{R}^n)^{N+1}$ and $R^i \subset [0, T]$ are measurable sets such that $\bigsqcup_{i=1}^K R^i = [0, T]$. Since λ is nonatomic, there exist $R_\rho^i \subset R^i$ such that $\lambda(R_\rho^i) = \rho\lambda(R^i)$ for all $i = 1, \dots, K$. Let us define $Q_\rho := \bigsqcup_{i=1}^K R_\rho^i \subset [0, T]$.

Note that $\lambda(Q_\rho) = \rho T$. Moreover, it holds that

$$\begin{aligned} & \left\| \int_0^T \left(1 - \frac{1}{\rho} \mathbf{1}_{Q_\rho}(s) \right) B(s) ds \right\|_{(\mathbb{R}^n)^{N+1}} \\ & \leq \left\| \int_0^T \left(1 - \frac{1}{\rho} \mathbf{1}_{Q_\rho}(s) \right) J(s) ds \right\|_{(\mathbb{R}^n)^{N+1}} + \left\| \int_0^T \left(1 - \frac{1}{\rho} \mathbf{1}_{Q_\rho}(s) \right) (B(s) - J(s)) ds \right\|_{(\mathbb{R}^n)^{N+1}}. \end{aligned}$$

The second integral in the left term can be easily bounded by $\frac{\rho}{2}$ and the first one is equal to

$$\left\| \sum_{i=1}^K a_i \int_0^T \left(\mathbf{1}_{R^i}(s) - \frac{1}{\rho} \mathbf{1}_{Q_\rho \cap R^i}(s) \right) ds \right\|_{(\mathbb{R}^n)^{N+1}} = \left\| \sum_{i=1}^K a_i \left(\lambda(R^i) - \frac{1}{\rho} \lambda(R_\rho^i) \right) \right\|_{(\mathbb{R}^n)^{N+1}} = 0.$$

The proof is complete. \square

Let us now conclude the proof of Lemma 3. Let $t \in [0, T]$. There exists $r \in \{0, \dots, N-1\}$ such that $t \in [t_r, t_{r+1}]$. In particular, it holds that $|t - t_r| < \delta$ and thus $\|b(t, \cdot) - b(t_r, \cdot)\|_{L^1} \leq \frac{\rho^2}{2(\rho+1)}$ (see remark after Lemma 5). It holds that

$$\begin{aligned} & \left\| \int_0^t \left(1 - \frac{1}{\rho} \mathbf{1}_{Q_\rho}(s) \right) h(s) ds \right\|_{\mathbb{R}^n} = \left\| \int_0^T \left(1 - \frac{1}{\rho} \mathbf{1}_{Q_\rho}(s) \right) b(t, s) ds \right\|_{\mathbb{R}^n} \\ & \leq \left\| \int_0^T \left(1 - \frac{1}{\rho} \mathbf{1}_{Q_\rho}(s) \right) (b(t, s) - b(t_r, s)) ds \right\|_{\mathbb{R}^n} + \left\| \int_0^T \left(1 - \frac{1}{\rho} \mathbf{1}_{Q_\rho}(s) \right) b(t_r, s) ds \right\|_{\mathbb{R}^n}. \end{aligned}$$

The first term can be bounded by $(1 + \frac{1}{\rho})\|b(t, \cdot) - b(t_r, \cdot)\|_{L^1} \leq \frac{\rho}{2}$ and the second one can be bounded by $\left\| \int_0^T \left(1 - \frac{1}{\rho} \mathbf{1}_{Q_\rho}(s) \right) B(s) ds \right\|_{(\mathbb{R}^n)^{N+1}} \leq \frac{\rho}{2}$ (see Lemma 6). Finally, we have proved that

$$\left\| \int_0^t \left(1 - \frac{1}{\rho} \mathbf{1}_{Q_\rho}(s) \right) h(s) ds \right\|_{\mathbb{R}^n} \leq \rho.$$

The proof of Lemma 3 is complete.

B Some recalls about Banach spaces geometry

B.1 Renorming a separable Banach space

Let $(X, \|\cdot\|)$ be a normed linear space. The *dual space* of $(X, \|\cdot\|)$ is $X^* := \mathcal{L}((X, \|\cdot\|), \mathbb{R})$ and X^* can be endowed with the *dual norm* of $\|\cdot\|$ defined by

$$\begin{aligned} \|\cdot\|_* : X^* &\longrightarrow \mathbb{R}^+ \\ f &\longmapsto \sup_{\substack{x \in X \\ \|x\| \leq 1}} |\langle f, x \rangle_{X^* \times X}|. \end{aligned}$$

In this case, we denote $(X^*, \|\cdot\|_*) = \text{dual}((X, \|\cdot\|))$. Recall that $(X^*, \|\cdot\|_*)$ is a Banach space, even if $(X, \|\cdot\|)$ is not.

Lemma 7. *Let $(X, \|\cdot\|)$ be a Banach space⁵ and $(X^*, \|\cdot\|_*) = \text{dual}((X, \|\cdot\|))$. Let N^* be a norm on X^* equivalent to $\|\cdot\|_*$. Then, the following properties are equivalent:*

⁵The Banach assumption is necessary in order to apply [4, Proposition 3.13 p.63] deriving from the classical Banach-Steinhaus theorem.

1. N^* is weak* lower semicontinuous on $(X^*, \|\cdot\|_*)$;

2. There exists N a norm on X equivalent to $\|\cdot\|$ such that $(X^*, N^*) = \text{dual}((X, N))$.

Remark 4. This lemma is given in [13, Lemma 3.94 p.251].

Proof. $2 \Rightarrow 1$. Since $(X, \|\cdot\|)$ is a Banach space and since N is equivalent to $\|\cdot\|$, note that (X, N) is a Banach space. Let us prove that N^* is weak* lower semicontinuous on $(X^*, \|\cdot\|_*)$. Let $(f_n) \subset X^*$ and $f \in X^*$. From [4, Proposition 3.13 p.63], (f_n) weak* converges to f in $(X^*, \|\cdot\|_*) = \text{dual}((X, \|\cdot\|))$ if and only if $\langle f_n, x \rangle_{X^* \times X}$ tends to $\langle f, x \rangle_{X^* \times X}$ for every $x \in X$ if and only if (f_n) weak* converges to f in $(X^*, N^*) = \text{dual}((X, N))$. Since (X, N) is a Banach space, we obtain from [4, Proposition 3.13 p.63] that $(N^*(f_n))$ is bounded and $N^*(f) \leq \liminf N^*(f_n)$.

$1 \Rightarrow 2$. We know that there exist $0 < \mu_1 \leq \mu_2$ such that $\mu_1 \|f\|_* \leq N^*(f) \leq \mu_2 \|f\|_*$ for every $f \in X^*$. Our proof is based on two steps.

First step - Definition of N and equivalence to $\|\cdot\|$. We define

$$\begin{aligned} N : X &\longrightarrow \mathbb{R}^+ \\ x &\longmapsto \sup_{\substack{f \in X^* \\ N^*(f) \leq 1}} |\langle f, x \rangle_{X^* \times X}|. \end{aligned}$$

First of all, let us note that N is well-defined since $|\langle f, x \rangle_{X^* \times X}| \leq \|f\|_* \|x\| \leq \frac{1}{\mu_1} N^*(f) \|x\| \leq \frac{1}{\mu_1} \|x\|$ for every $x \in X$ and every $f \in X^*$ such that $N^*(f) \leq 1$. In particular it holds that $N(x) \leq \frac{1}{\mu_1} \|x\|$ for every $x \in X$. Let us prove that N is a norm on X . Clearly we have $N(\lambda x) = |\lambda| N(x)$ and $N(x + y) \leq N(x) + N(y)$ for every $x, y \in X$ and every $\lambda \in \mathbb{R}$. Moreover, it holds that $N(0) = 0$. Now let us consider $x \neq 0$. From the classical Hahn-Banach theorem (see [4, Corollary 1.3 p.3]) applied to $(X, \|\cdot\|)$ (with dual $(X^*, \|\cdot\|_*)$), there exists $f \in X^*$ such that $\langle f, x \rangle_{X^* \times X} = \|x\|^2$ and $\|f\|_* = \|x\|$. Let $g := \frac{f}{\mu_2 \|x\|} \in X^*$. It holds that $N^*(g) = \frac{N^*(f)}{\mu_2 \|x\|} \leq \frac{\mu_2 \|f\|_*}{\mu_2 \|x\|} = 1$ and $|\langle g, x \rangle_{X^* \times X}| = \frac{|\langle f, x \rangle_{X^* \times X}|}{\mu_2 \|x\|} = \frac{\|x\|^2}{\mu_2 \|x\|} = \frac{\|x\|}{\mu_2}$. Thus $N(x) \geq \frac{\|x\|}{\mu_2} > 0$. As a consequence, we have proved that $N(x) = 0$ if and only if $x = 0$. We conclude that N is a norm on X . Moreover, we have also proved that $\frac{1}{\mu_2} \|x\| \leq N(x) \leq \frac{1}{\mu_1} \|x\|$ for every $x \in X$ (the case $x = 0$ is obvious). As a consequence, N is equivalent to $\|\cdot\|$.

Second step - $(X^, N^*) = \text{dual}((X, N))$.* Since N is equivalent to $\|\cdot\|$, $\mathcal{L}((X, N), \mathbb{R}) = \mathcal{L}((X, \|\cdot\|), \mathbb{R}) = X^*$, i.e. X^* is the dual space of (X, N) . Let us introduce \tilde{N} the dual norm of N on X^* given by

$$\begin{aligned} \tilde{N} : X^* &\longrightarrow \mathbb{R}^+ \\ f &\longmapsto \sup_{\substack{x \in X \\ N(x) \leq 1}} |\langle f, x \rangle_{X^* \times X}|. \end{aligned}$$

In particular, we have $(X^*, \tilde{N}) = \text{dual}((X, N))$. Our aim is to prove that $N^* = \tilde{N}$. Firstly, let $f \in X^*$ such that $f \neq 0$. Let $x \in X$ such that $N(x) \leq 1$. From the definition of N , since $N^*(\frac{f}{N^*(f)}) = 1$, we have $|\langle \frac{f}{N^*(f)}, x \rangle_{X^* \times X}| \leq N(x) \leq 1$. Thus, $|\langle f, x \rangle_{X^* \times X}| \leq N^*(f)$ for every $x \in X$ such that $N(x) \leq 1$. As a consequence, from the definition of \tilde{N} , we obtain that $\tilde{N}(f) \leq N^*(f)$ for every $f \in X^*$ (the case $f = 0$ is obvious). Secondly, let us assume by contradiction that there exists $f_0 \in X^*$ such that $\tilde{N}(f_0) < N^*(f_0)$. Thus $f_0 \neq 0$ and we can define $g := \frac{f_0}{\tilde{N}(f_0)} \in X^*$ satisfying $\tilde{N}(g) = 1 < N^*(g)$. Hence $g \notin \overline{B}_{(X^*, N^*)}(0, 1)$. From Hypothesis 1, one can easily see that $\overline{B}_{(X^*, N^*)}(0, 1)$ is a nonempty weakly* closed convex of $(X^*, \|\cdot\|_*)$. Then, recall

that the dual of $(X^*, \|\cdot\|_*) = \text{dual}((X, \|\cdot\|))$ endowed with the classical weak* topology is the set $(\langle \cdot, x \rangle_{X^* \times X})_{x \in X}$ (see [4, Proposition 3.14 p.64]). Finally, from the classical Hahn-Banach theorem⁶ (see [4, Theorem 1.7 p.7]), there exists $x \in X$, $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ such that $\langle f, x \rangle_{X^* \times X} \leq \alpha - \varepsilon < \alpha + \varepsilon \leq \langle g, x \rangle_{X^* \times X}$ for every $f \in \overline{B}_{(X^*, N^*)}(0, 1)$. As a consequence, from the definition of N , we obtain that $N(x) < \langle g, x \rangle_{X^* \times X} \leq \tilde{N}(g)N(x)$ and finally $\tilde{N}(g) > 1$. This is a contradiction. \square

Proposition 4. *Let $(X, \|\cdot\|)$ be a separable Banach space and $(X^*, \|\cdot\|_*) = \text{dual}((X, \|\cdot\|))$. There exists a norm N on X equivalent to $\|\cdot\|$ such that:*

1. N^* is equivalent on X^* to $\|\cdot\|_*$;
2. N^* is strictly convex on X^* ;

where $(X^*, N^*) = \text{dual}((X, N))$.

Remark 5. This proposition is given in [11, Theorem 2.18 p.42].

Proof. Our proof is based on four steps, including the application of Lemma 7.

First step (definition of N^ and equivalence to $\|\cdot\|_*$).* Let $(e_k)_{k \in \mathbb{N}^*} \subset X$ be a sequence dense in $\overline{B}_{(X, \|\cdot\|)}(0, 1)$. Then, let us consider the linear operator given by

$$\begin{aligned} A : X^* &\longrightarrow \ell^2(\mathbb{N}^*, \mathbb{R}) \\ f &\longmapsto \left(\frac{1}{2^{k/2}} \langle f, e_k \rangle_{X^* \times X} \right)_{k \in \mathbb{N}^*}. \end{aligned}$$

Let us note that

$$\|A(f)\|_{\ell^2}^2 = \sum_{k \in \mathbb{N}^*} \frac{1}{2^k} |\langle f, e_k \rangle_{X^* \times X}|^2 \leq \|f\|_*^2,$$

for every $f \in X^*$. Now we define

$$\begin{aligned} N^* : X^* &\longrightarrow \mathbb{R}^+ \\ f &\longmapsto \|f\|_* + \|A(f)\|_{\ell^2}. \end{aligned}$$

Clearly N^* is a norm on X^* that is equivalent to $\|\cdot\|_*$ since $\|f\|_* \leq N^*(f) \leq 2\|f\|_*$ for every $f \in X^*$.

Second step (strict convexity of N^).* Let $f, g \in X^*$ and let $\lambda \in (0, 1)$ such that $N^*(\lambda f + (1 - \lambda)g) = \lambda N^*(f) + (1 - \lambda)N^*(g)$. Our aim is to prove that $f = \mu g$ for some $\mu \geq 0$. Since $\|\cdot\|_*$ and $\|A(\cdot)\|_{\ell^2}$ are convex, we get that $\|A(\lambda f + (1 - \lambda)g)\|_{\ell^2} = \lambda\|A(f)\|_{\ell^2} + (1 - \lambda)\|A(g)\|_{\ell^2}$. Computing the square of the previous equality and using the linearity of A and the bilinearity of $\langle \cdot, \cdot \rangle_{\ell^2}$, one can easily obtain that $\langle A(f), A(g) \rangle_{\ell^2} = \|A(f)\|_{\ell^2}\|A(g)\|_{\ell^2}$. From the classical Cauchy-Schwarz inequality in $\ell^2(\mathbb{N}^*, \mathbb{R})$, there exists $\mu \geq 0$ such that $A(f) = \mu A(g)$, that is, $A(f - \mu g) = 0$ and $\langle f - \mu g, e_k \rangle_{X^* \times X} = 0$ for every $k \in \mathbb{N}^*$. From density and homogeneity, we easily obtain that $\langle f - \mu g, x \rangle_{X^* \times X} = 0$ for every $x \in X$, i.e. $f - \mu g = 0$.

Third step (N^ is weak* lower semicontinuous on $(X^*, \|\cdot\|_*)$).* Let $(f_n) \subset X^*$ and $f \in X^*$ such that (f_n) weak* converges to f in $(X^*, \|\cdot\|_*) = \text{dual}((X, \|\cdot\|))$, i.e. $\langle f_n, x \rangle_{X^* \times X}$ tends to $\langle f, x \rangle_{X^* \times X}$ for every $x \in X$ (see [4, Proposition 3.13 p.63]). Let us prove that $N^*(f) \leq \liminf N^*(f_n)$. First of all, recall that $(\|f_n\|_*)$ is bounded by some $M \geq 0$ and that $\|f\|_* \leq \liminf \|f_n\|_*$ (see [4, Proposition 3.13 p.63]). Thus, $\liminf N^*(f_n) \geq \liminf \|f_n\|_* + \liminf \|A(f_n)\|_{\ell^2} \geq \|f\|_* +$

⁶The classical Hahn-Banach theorem can be applied here since $(X^*, \|\cdot\|_*) = \text{dual}((X, \|\cdot\|))$ endowed with the classical weak* topology is a topological vector space that is locally convex.

$\liminf \|A(f_n)\|_{\ell^2}$. Let us recall that $\|A(f_n)\|_{\ell^2}^2 = \sum_{k \in \mathbb{N}^*} \frac{1}{2^k} \langle f_n, e_k \rangle_{X^* \times X}^2$. Since $(\frac{1}{2^k} \langle f_n, e_k \rangle_{X^* \times X}^2)$ converge to $\frac{1}{2^k} \langle f, e_k \rangle_{X^* \times X}^2$ and since $|\frac{1}{2^k} \langle f_n, e_k \rangle_{X^* \times X}^2| \leq \frac{1}{2^k} \|f_n\|_*^2 \|e_k\|^2 \leq \frac{M^2}{2^k} \in \ell^1(\mathbb{N}^*, \mathbb{R})$ for every $n \in \mathbb{N}$, we obtain from the classical Lebesgue dominated convergence theorem that $\lim \|A(f_n)\|_{\ell^2} = \|A(f)\|_{\ell^2}$. Finally, we have proved that $\liminf N^*(f_n) \geq \|f\|_* + \|A(f)\|_{\ell^2} = N^*(f)$.

Fourth step (conclusion). We conclude the proof by applying Lemma 7. \square

B.2 The distance function

For this section, we essentially refer to [12, 13].

In the whole section, $(X, \|\cdot\|)$ denotes a normed linear space and $(X^*, \|\cdot\|_*) := \text{dual}((X, \|\cdot\|))$. In this section, we denote by $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{X^* \times X}$.

Lemma 8. *Let $S \subset X$ be a convex subset such that $\text{Int}(S) \neq \emptyset$. Then, $\text{Adh}(\text{Int}(S)) = \text{Adh}(S)$.*

Proof. Let us prove that $\text{Adh}(S) \subset \text{Adh}(\text{Int}(S))$. Let $x \in \text{Adh}(S)$ and let $y \in \text{Int}(S)$ such that $y \neq x$. There exists $\varepsilon > 0$ such that $y + B_X(0, \varepsilon) \subset S$. Let us prove that the line segment $(x, y) \subset \text{Int}(S)$. Let $w := tx + (1-t)y$ for some $t \in (0, 1)$. Since $x \in \text{Adh}(S)$, there exists $x' \in S$ such that $\|x - x'\| < \varepsilon \frac{1-t}{t}$. Let us denote by $z := x - x'$. Finally, we have $w = tx' + (1-t)y + tz = tx' + (1-t)(y + \xi)$ where $\xi := \frac{t}{1-t}z \in B_X(0, \varepsilon)$. As a consequence, since $x' \in S$, since $y + B_X(0, \varepsilon) \subset S$ and since S is convex, $w \in tx' + (1-t)(y + B_X(0, \varepsilon)) \subset S$, that is, w belongs to an open subset included in S , i.e. $w \in \text{Int}(S)$. Finally, we have proved that $(x, y) \subset \text{Int}(S)$ and thus $x \in \text{Adh}(\text{Int}(S))$. \square

Let $\varphi : X \rightarrow \mathbb{R}$ be a convex function and $x \in X$. Recall that the subdifferential of φ at x is defined by

$$\partial\varphi(x) := \{f \in X^* \mid \langle f, y - x \rangle \leq \varphi(y) - \varphi(x), \forall y \in X\}.$$

Note that $\partial\varphi(x)$ is a weakly* closed convex subset of X^* .

Lemma 9. *If $\varphi : X \rightarrow \mathbb{R}$ is continuous, then $\partial\varphi(x) \neq \emptyset$ for every $x \in X$.*

Proof. Recall that the epigraph of φ is defined by

$$\text{Epi}_\varphi := \{(y, \lambda) \in X \times \mathbb{R} \mid \varphi(y) \leq \lambda\} \subset X \times \mathbb{R}.$$

Since φ is convex and continuous on X , Epi_φ is clearly a nonempty closed convex subset of $X \times \mathbb{R}$. Let $x \in X$. It is clear that $(x, \varphi(x)) \notin \text{Int}(\text{Epi}_\varphi)$ (since $(x, \varphi(x) - \varepsilon) \notin \text{Epi}_\varphi$ for all $\varepsilon > 0$) and that $\text{Int}(\text{Epi}_\varphi)$ is a nonempty⁷ open convex⁸ subset of $X \times \mathbb{R}$. From the classical Hahn-Banach theorem (see [4, Lemma 1.3 p.6]), there exists $(f, c) \in (X \times \mathbb{R})^* = X^* \times \mathbb{R}$ such that $\langle (f, c), (x, \varphi(x)) \rangle < \langle (f, c), (w, r) \rangle$, i.e. $\langle f, w - x \rangle + c(r - \varphi(x)) > 0$ for all $(w, r) \in \text{Int}(\text{Epi}_\varphi)$. From the continuity of φ at x , it follows that $(x, \varphi(x) + 1) \in \text{Int}(\text{Epi}_\varphi)$. Thus, taking $(w, r) = (x, \varphi(x) + 1)$, we obtain that $c > 0$. From Lemma 8, we also obtain that $\langle f, w - x \rangle + c(r - \varphi(x)) \geq 0$ for all $(w, r) \in \text{Epi}_\varphi$. Finally, taking $r = \varphi(w)$, we conclude that $\langle -\frac{1}{c}f, w - x \rangle \leq \varphi(w) - \varphi(x)$ for all $w \in X$, i.e. $-\frac{1}{c}f \in \partial\varphi(x)$. The proof is complete. \square

Lemma 10. *If $\varphi : X \rightarrow \mathbb{R}$ is L -Lipschitz continuous on X for some $L \geq 0$, then $\partial\varphi(x) \neq \emptyset$ and $\partial\varphi(x) \subset \overline{B}_{X^*}(0, L)$ for every $x \in X$.*

Proof. Let $x \in X$ and $f \in \partial\varphi(x) \neq \emptyset$ (see Lemma 9). One can easily obtain that $|\langle f, z \rangle| = \langle f, \pm z \rangle = \langle f, \pm z + x - x \rangle \leq \varphi(\pm z + x) - \varphi(x) \leq |\varphi(\pm z + x) - \varphi(x)| \leq L\|\pm z + x - x\| = L\|z\|$ for every $z \in X$. Thus $\|f\|_* \leq L$. \square

⁷ $\text{Int}(\text{Epi}_\varphi)$ is nonempty since, from Lemma 8, $\text{Adh}(\text{Int}(\text{Epi}_\varphi)) = \text{Adh}(\text{Epi}_\varphi) = \text{Epi}_\varphi$ is nonempty.

⁸One can easily prove that the interior of a convex set is a convex set.

Lemma 11. *If $\varphi : X \rightarrow \mathbb{R}$ is Gâteaux-differentiable at some $x \in X$, then $\partial\varphi(x) = \{D\varphi(x)\}$.*

Proof. Firstly, let us prove that $D\varphi(x) \in \partial\varphi(x)$. From the convexity of φ , it holds that $\varphi(x + \lambda(y - x)) \leq \varphi(x) + \lambda(\varphi(y) - \varphi(x))$, i.e. $\frac{\varphi(x + \lambda(y - x)) - \varphi(x)}{\lambda} \leq \varphi(y) - \varphi(x)$ for any $y \in X$ and any $\lambda \in (0, 1)$. Passing to the limit $\lambda \rightarrow 0^+$, we obtain that $\langle D\varphi(x), y - x \rangle \leq \varphi(y) - \varphi(x)$ for every $y \in X$. Thus $D\varphi(x) \in \partial\varphi(x)$. Secondly, let us prove that $\partial\varphi(x) \subset \{D\varphi(x)\}$. Let $f \in \partial\varphi(x)$. It holds that $\langle f, \lambda(y - x) \rangle \leq \varphi(x + \lambda(y - x)) - \varphi(x)$, i.e. $\langle f, y - x \rangle \leq \frac{\varphi(x + \lambda(y - x)) - \varphi(x)}{\lambda}$ for every $y \in X$ and $\lambda \in (0, 1)$. Passing to the limit $\lambda \rightarrow 0^+$, we get that $\langle f, y - x \rangle \leq \langle D\varphi(x), y - x \rangle$ for every $y \in X$. From linearity, we conclude that $\langle f, y \rangle = \langle D\varphi(x), y \rangle$ for all $y \in X$, i.e. $f = D\varphi(x)$. \square

Lemma 12. *If $\varphi : X \rightarrow \mathbb{R}$ is Lipschitz continuous around $x \in X$ and if $\partial\varphi(x)$ is reduced to a singleton $\{f\}$, then φ is strictly Hadamard-differentiable at x with $D\varphi(x) = f$.*

Proof. See [12, Theorem 3.54 p.313]. \square

Lemma 13. *Let $S \subset X$ be a nonempty subset and let $d_S : X \rightarrow \mathbb{R}$ be the distance function to S . The following properties hold:*

1. d_S is 1-Lipschitz continuous;
2. If S is convex, d_S is convex;
3. If S is closed and convex, $\partial d_S(x) \neq \emptyset$ and $\partial d_S(x) \subset \text{Sph}_{X^*}(0, 1)$ for every $x \in X \setminus S$.

Proof. 1. Let $x, y \in S$. Let $(y_n) \subset S$ be a sequence such that $\|y - y_n\| \rightarrow d_S(y)$. Then it holds that $d_S(x) \leq \|x - y_n\| \leq \|x - y\| + \|y - y_n\| \rightarrow \|x - y\| + d_S(y)$. In a very similar way we obtain that $d_S(y) \leq \|x - y\| + d_S(x)$. We conclude that $|d_S(y) - d_S(x)| \leq \|y - x\|$.

2. Let $x, y \in S$ and $\lambda \in [0, 1]$. Let $(x_n), (y_n) \subset S$ such that $\|x - x_n\| \rightarrow d_S(x)$ and $\|y - y_n\| \rightarrow d_S(y)$. Note that $(1 - \lambda)x_n + \lambda y_n \in S$ since S is convex. Then, $d_S((1 - \lambda)x + \lambda y) \leq \|[(1 - \lambda)x + \lambda y] - [(1 - \lambda)x_n + \lambda y_n]\| \leq (1 - \lambda)\|x - x_n\| + \lambda\|y - y_n\| \rightarrow (1 - \lambda)d_S(x) + \lambda d_S(y)$. Thus, d_S is convex.

3. Let $x \in X \setminus S$ and $f \in \partial d_S(x) \neq \emptyset$ (see Lemma 10). From Lemma 10, we already know that $\|f\|_* \leq 1$. Since S is closed, note that $d_S(x) > 0$. There exists $(x_n) \subset S$ such that $\|x - x_n\| \rightarrow d_S(x)$. Moreover, it holds that $\langle f, x_n - x \rangle \leq d_S(x_n) - d_S(x) = -d_S(x)$. Thus, $d_S(x) \leq \langle f, x - x_n \rangle \leq \|f\|_* \|x - x_n\| \rightarrow \|f\|_* d_S(x)$. Thus, $\|f\|_* \geq 1$. We conclude that $\partial d_S(x) \subset \text{Sph}_{X^*}(0, 1)$. \square

Remark 6. Since d_S is 1-Lipschitz continuous on X , one can easily prove that d_S^2 is Fréchet-differentiable on S , with $Dd_S^2(x) = 0$ for every $x \in S$.

Proposition 5. *Let us assume that $\|\cdot\|_*$ is strictly convex on X^* . Let $S \subset X$ be a nonempty closed and convex subset. Then, d_S is strictly Hadamard-differentiable on $X \setminus S$ with $\|Dd_S(x)\|_* = 1$ for every $x \in X \setminus S$.*

Proof. Let $x \in X \setminus S$. From Lemma 13, it follows that $\partial d_S(x) \neq \emptyset$ and $\partial d_S(x) \subset \text{Sph}_{X^*}(0, 1)$. Since $\partial d_S(x) \neq \emptyset$ is a convex subset included in $\text{Sph}_{X^*}(0, 1)$ and since $\|\cdot\|_*$ is strictly convex, it follows that $\partial d_S(x)$ is necessarily reduced to a singleton $\{f\}$ satisfying $\|f\|_* = 1$. We conclude from Lemma 12. \square

C Functions of bounded variations and Stieltjes integrals

For this section, we essentially refer to [2, 5, 6, 7, 10, 15].

C.1 Some recalls

Recall that a function $\eta : [0, T] \rightarrow \mathbb{R}$ is said to be of *bounded variations* if

$$V(\eta) := \sup_{(t_k)_k} \left\{ \sum_k |\eta(t_{k+1}) - \eta(t_k)| \right\} < +\infty$$

where the supremum is taken over all partitions $(t_k)_k$ of $[0, T]$. In that case, we denote by $\eta \in BV_1$.

Recall that $\eta : [0, T] \rightarrow \mathbb{R}$ belongs to BV_1 if and only if η is equal to the difference of two monotonically increasing functions. In particular, if $\eta \in BV_1$, then η admits a left-limit denoted by $\eta(t^-)$ (resp. a right-limit denoted by $\eta(t^+)$) at every point $t \in (0, T]$ (resp. $t \in [0, T)$) and the set of discontinuity points of η is at most countable.

Recall that the application $\|\cdot\|_{BV_1}$ given by

$$\begin{aligned} \|\cdot\|_{BV_1} : BV_1 &\longrightarrow \mathbb{R}^+ \\ \eta &\longmapsto \eta(0) + V(\eta) \end{aligned}$$

defines a norm on BV_1 . Moreover, recall that $(BV_1, \|\cdot\|_{BV_1})$ is a Banach space.

Let $\eta \in BV_1$. For every $t \in [0, T]$, $|\eta(t)| \leq |\eta(0)| + |\eta(t) - \eta(0)| \leq |\eta(0)| + |\eta(t) - \eta(0)| + |\eta(T) - \eta(t)| \leq |\eta(0)| + V(\eta) \leq \|\eta\|_{BV_1}$. As a consequence, $\eta \in BF_1$ with $\|\eta\|_\infty \leq \|\eta\|_{BV_1}$.

Recall that if $z \in C_1$ and $\eta \in BV_1$, then the classical Riemann-Stieltjes integral defined by

$$\int_0^T z(\tau) d\eta(\tau) := \lim \sum_k z(t_k)(\eta(t_{k+1}) - \eta(t_k))$$

exists. In the above equality, the limit means that the length of the partition $(t_k)_k$ tends to zero.

Let us recall the following classical Riesz theorem.

Proposition 6 (Riesz theorem). *Let $\varphi \in C_1^*$. There exists a unique $\eta \in NBV_1$ such that*

$$\langle \varphi, z \rangle_{C_1^* \times C_1} = \int_0^T z(\tau) d\eta(\tau),$$

for every $z \in C_1$. Moreover, if $\langle \varphi, z \rangle_{C_1^ \times C_1} \geq 0$ for every $z \in C_1^+$, then η is monotonically increasing.*

We refer to [10, p.245] for a complete proof of Proposition 6. In this section we will only detail the proof of the following weaker result, which is sufficient for the completeness of this note.

Proposition 7 (Riesz corollary). *Let $\varphi \in C_1^*$ such that $\langle \varphi, z \rangle_{C_1^* \times C_1} \geq 0$ for every $z \in C_1^+$. There exists $\eta \in NBV_1$ such that η is monotonically increasing on $[0, T]$ and*

$$\langle \varphi, z \rangle_{C_1^* \times C_1} = \int_0^T z(\tau) d\eta(\tau),$$

for every $z \in C_1$. Moreover, from construction of η , $\varphi = 0$ if and only if $\eta = 0$.

Proof. If $\varphi = 0$, it is sufficient to consider $\eta = 0 \in NBV_1$ that is monotonically increasing. Now let us consider that $\varphi \neq 0$. In the sequel we simply denote by $1 \in C_1^+ \subset BF_1^+$ the constant function

equal to 1 on $[0, T]$. Finally we denote by $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{C_1^* \times C_1}$. When no confusion is possible, $\langle \cdot, \cdot \rangle$ also denotes $\langle \cdot, \cdot \rangle_{BF_1^* \times BF_1}$.

Since $\langle \varphi, z \rangle \geq 0$ for every $z \in C_1^+$, it holds that $1 - \xi \in C_1^+$ and then $\langle \varphi, \xi \rangle \leq \langle \varphi, 1 \rangle$ for every $\xi \in C_1$ such that $\|\xi\|_\infty \leq 1$. We deduce that $\|\varphi\|_{C_1^*} \leq \langle \varphi, 1 \rangle \leq \|\varphi\|_{C_1^*} \|1\|_\infty = \|\varphi\|_{C_1^*}$ and then $\|\varphi\|_{C_1^*} = \langle \varphi, 1 \rangle$. From the classical Hahn-Banach theorem (see [4, Corollary 1.2 p.3]), there exists a linear continuous application $\tilde{\varphi} : BF_1 \rightarrow \mathbb{R}$ such that $\tilde{\varphi}$ extends φ to BF_1 and $\|\tilde{\varphi}\|_{BF_1^*} = \|\varphi\|_{C_1^*}$. In particular it holds that $\|\tilde{\varphi}\|_{BF_1^*} = \langle \tilde{\varphi}, 1 \rangle$. Let us prove that $\langle \tilde{\varphi}, z \rangle \geq 0$ for every $z \in BF_1^+$. Let $z \in BF_1^+$ such that $z \neq 0$ and let $\xi := \frac{2}{\|z\|_\infty} z - 1$. Then $\xi \in BF_1$ with $\|\xi\|_\infty \leq 1$. As a consequence $-\langle \tilde{\varphi}, \xi \rangle \leq |\langle \tilde{\varphi}, \xi \rangle| \leq \|\tilde{\varphi}\|_{BF_1^*} \|\xi\|_\infty \leq \langle \tilde{\varphi}, 1 \rangle$. Finally, it holds that $\langle \tilde{\varphi}, z \rangle = \frac{\|z\|_\infty}{2} (\langle \tilde{\varphi}, \xi \rangle + \langle \varphi, 1 \rangle) \geq 0$.

Now we introduce $\eta(t) := \langle \tilde{\varphi}, \mathbf{1}_{(0,t]} \rangle$ for every $t \in [0, T]$. In particular $\eta(0) = 0$. Moreover, since $\mathbf{1}_{(0,t]} - \mathbf{1}_{(0,s]} \in BF_1^+$ for every $0 \leq s \leq t \leq T$ and since $\langle \tilde{\varphi}, z \rangle \geq 0$ for every $z \in BF_1^+$, it follows that η is monotonically increasing on $[0, T]$. In particular $\eta \in BV_1$ with $V(\eta) = \eta(T) - \eta(0)$.

Now let us prove that $\langle \varphi, z \rangle = \int_0^T z(\tau) d\eta(\tau)$ for every $z \in C_1$. Let $z \in C_1$ and let $\varepsilon > 0$. Since z is uniformly continuous on $[0, T]$, there exists $\delta > 0$ such that

$$\|\varphi\|_{C_1^*} |z(t) - z(s)| \leq \frac{\varepsilon}{2}$$

for every $(t, s) \in [0, T]^2$ such that $|t - s| \leq \delta$. Let $(t_k)_k$ be a partition of $[0, T]$ such that $t_{k+1} - t_k \leq \delta$ and such that

$$\left| \int_0^T z(\tau) d\eta(\tau) - \sum_k z(t_k) (\eta(t_{k+1}) - \eta(t_k)) \right| \leq \frac{\varepsilon}{2}.$$

Then, we introduce $u = \sum_k z(t_k) \mathbf{1}_{(t_k, t_{k+1}]} \in BF_1$. It clearly holds that

$$\|\varphi\|_{C_1^*} \|z - u\|_\infty \leq \frac{\varepsilon}{2} \quad \text{and} \quad \langle \tilde{\varphi}, u \rangle = \sum_k z(t_k) (\eta(t_{k+1}) - \eta(t_k)).$$

As a consequence it holds that

$$\left| \int_0^T z(\tau) d\eta(\tau) - \langle \varphi, z \rangle \right| \leq \left| \int_0^T z(\tau) d\eta(\tau) - \langle \tilde{\varphi}, u \rangle \right| + |\langle \tilde{\varphi}, u \rangle - \langle \varphi, z \rangle| \leq \varepsilon.$$

In particular, it holds that $\eta(T) = \eta(T) - \eta(0) = \int_0^T 1 d\eta(\tau) = \langle \varphi, 1 \rangle = \|\varphi\|_{C_1^*} \neq 0$. Thus $\eta \neq 0$.

To conclude the proof, let us prove that η can be chosen left-continuous on $(0, T)$. First of all, since $\eta(0) = 0$ and η is monotonically increasing on $[0, T]$, $\eta(t^-)$ exists for all $0 < t \leq T$ and it holds that

$$0 \leq \eta(s) \leq \eta(t^-) \leq \eta(t) \leq \eta(T), \tag{15}$$

for every $0 \leq s < t \leq T$. Let us define

$$\nu(t) := \begin{cases} 0 & \text{if } t = 0, \\ \eta(t^-) & \text{if } 0 < t < T, \\ \eta(T) & \text{if } t = T. \end{cases}$$

In particular $\nu \neq 0$. Using (15), one can easily prove that ν is monotonically increasing on $[0, T]$. Let us assume by contradiction that ν is not left-continuous on $(0, T)$, that is, there exists $t \in (0, T)$

and $\varepsilon > 0$ such that $\nu(t^-) \leq \nu(t) - \varepsilon$. Finally, from (15), we obtain that

$$\eta\left(t - \frac{2}{k}\right) = \eta\left(t - \frac{1}{k} - \frac{1}{k}\right) \leq \eta\left(\left(t - \frac{1}{k}\right)^-\right) = \nu\left(t - \frac{1}{k}\right) \leq \nu(t^-) \leq \nu(t) - \varepsilon = \eta(t^-) - \varepsilon,$$

for every $k \in \mathbb{N}^*$. This raises a contradiction when k tends to $+\infty$. Hence $\nu \in \text{NBV}_1$. Now let us prove that

$$\int_0^T z(\tau) d\eta(\tau) = \int_0^T z(\tau) d\nu(\tau)$$

for every $z \in C_1$. Since η and ν can be different only at discontinuity points of η (which are at most countable), we consider a sequence of partitions $((t_k^\ell)_k)_\ell$ of $[0, T]$ such that the length of the partitions tends to zero when ℓ tends to $+\infty$ and such that no point t_k^ℓ is a discontinuity point of η . As a consequence, it holds that

$$\int_0^T z(\tau) d\eta(\tau) = \lim_{\ell \rightarrow \infty} \sum_k z(t_k^\ell)(\eta(t_{k+1}^\ell) - \eta(t_k^\ell)) = \lim_{\ell \rightarrow \infty} \sum_k z(t_k^\ell)(\nu(t_{k+1}^\ell) - \nu(t_k^\ell)) = \int_0^T z(\tau) d\nu(\tau)$$

for every $z \in C_1$. \square

In Appendix D.3, we will need the two following results.

Lemma 14. *If $h \in \text{AC}_1$, then $h \in \text{BV}_1$.*

Proof. One can easily get that

$$\sum_k |h(t_{k+1}) - h(t_k)| = \sum_k \left| \int_{t_k}^{t_{k+1}} \dot{h}(\tau) d\tau \right| \leq \int_0^T |\dot{h}(\tau)| d\tau = \|\dot{h}\|_{L^1},$$

for every partition $(t_k)_k$ of $[0, T]$. \square

Lemma 15. *Let $\eta \in \text{NBV}_1$ be monotonically increasing and $z \in C_1$. The function $h : [0, T] \rightarrow \mathbb{R}$ defined by*

$$h(t) := \int_0^t z(\tau) d\eta(\tau),$$

is of bounded variations, i.e. $h \in \text{BV}_1$.

Proof. One can easily get that

$$\sum_k |h(t_{k+1}) - h(t_k)| = \sum_k \left| \int_{t_k}^{t_{k+1}} z(\tau) d\eta(\tau) \right| \leq \sum_k \int_{t_k}^{t_{k+1}} |z(\tau)| d\eta(\tau) \leq \int_0^T |z(\tau)| d\eta(\tau),$$

for every partition $(t_k)_k$ of $[0, T]$. \square

We conclude this section with some recalls about the Lebesgue-Stieltjes integral. Let $\eta \in \text{NBV}_1$ be monotonically increasing. Then η induces a finite nonnegative measure on the Borel set of $[0, T]$ denoted by $d\eta$. This measure is constructed from the equalities $d\eta([a, b)) = \eta(b) - \eta(a)$ for every $0 \leq a \leq b \leq T$ and extended from the classical Carathéodory extension theorem. For every $z \in C_1$, the Riemann-Stieltjes integral of z with respect to η and the Lebesgue-Stieltjes integral of z with respect to $d\eta$ (that corresponds to the classical Lebesgue integral of z with respect to the measure $d\eta$) coincide. We refer to [7, p.83] or [15, p.288] for more details. Finally, the following Fubini-type formula holds:

$$\int_0^T \int_0^\tau z(\tau, s) ds d\eta(\tau) = \int_0^T \int_s^T z(\tau, s) d\eta(\tau) ds \quad (16)$$

for every $z \in L^\infty([0, T]^2, \mathbb{R})$ such that z is continuous in its first variable.

Remark 7. Note that the Stieltjes integrals in (16) are both well-defined in terms of Riemann-Stieltjes integration⁹ and the classical integrals in (16) have to be understood in the Lebesgue sense. Actually, one can easily see that the double integrals in (16) both exist.

C.2 Notations and Fubini-type formulas

For any $\eta = (\eta_i)_{i=1,\dots,j} \in \text{NBV}_j$ such that η_i is monotonically increasing and for any $z = (z_i)_{i=1,\dots,j} \in C_j$, we denote by

$$\int_0^T \langle z(\tau), d\eta(\tau) \rangle := \sum_{i=1}^j \int_0^T z_i(\tau) d\eta_i(\tau) \in \mathbb{R}.$$

Let $r \in \mathbb{N}^*$. We denote by

$$\int_0^T A(\tau) \times d\eta(\tau) := \left(\sum_{i=1}^j \int_0^T a_{ki}(\tau) d\eta_i(\tau) \right)_{k=1,\dots,r} \in \mathbb{R}^r,$$

and

$$\int_0^T \langle z(\tau), A(\tau) \times d\eta(\tau) \rangle := \int_0^T \langle A(\tau)^\top \times z(\tau), d\eta(\tau) \rangle \in \mathbb{R},$$

for every continuous matrices $A(\cdot) = (a_{ki}(\cdot))_{ki} : [0, T] \rightarrow \mathbb{R}^{r,j}$ and every $z \in C_r$. Moreover, one can easily prove that if $z \in \mathbb{R}^r$ (i.e. $z \in C_r$ constant) then

$$\int_0^T \langle z, A(\tau) \times d\eta(\tau) \rangle = \left\langle z, \int_0^T A(\tau) \times d\eta(\tau) \right\rangle_{\mathbb{R}^r \times \mathbb{R}^r}. \quad (17)$$

Finally, one can prove from Equality (16) that the following Fubini-type formulas both hold:

$$\int_0^T \left\langle \int_0^\tau \Phi(\tau, s) ds, d\eta(\tau) \right\rangle = \int_0^T \int_s^T \langle \Phi(\tau, s), d\eta(\tau) \rangle ds, \quad (18)$$

and

$$\int_0^T \left(\int_0^\tau A(\tau, s) ds \right) \times d\eta(\tau) = \int_0^T \left(\int_s^T A(\tau, s) \times d\eta(\tau) \right) ds, \quad (19)$$

where $\Phi \in L^\infty([0, T]^2, \mathbb{R}^j)$ and $A \in L^\infty([0, T]^2, \mathbb{R}^{r,j})$ are continuous in their first variable.

D State-transition matrices and linear Cauchy-Stieltjes problems

In the whole section $A \in L^\infty([0, T], \mathbb{R}^{n,n})$.

⁹Indeed, one can easily prove from the classical Lebesgue dominated convergence theorem that the function $\tau \mapsto \int_0^\tau z(\tau, s) ds$ is continuous on $[0, T]$.

D.1 Recalls on state-transition matrices

For every $s \in [0, T]$, the backward/forward linear Cauchy problem $(\text{BFCP}_{A,s})$ given by

$$\begin{cases} \dot{Z}(t) = A(t) \times Z(t), & \text{a.e. } t \in [0, T], \\ Z(s) = \text{Id}_n, \end{cases} \quad (\text{BFCP}_{A,s})$$

admits a unique maximal solution that is moreover global.¹⁰ We denote this solution by $Z(\cdot, s) : [0, T] \rightarrow \mathbb{R}^{n,n}$. The matrix function $Z(\cdot, \cdot)$ is the so-called *state-transition matrix* associated to A .

Lemma 16. *The following equalities both hold*

$$\begin{aligned} Z(t, s) &= \text{Id}_n + \int_s^t A(\tau) \times Z(\tau, s) d\tau, \\ &= \text{Id}_n + \int_s^t Z(t, \tau) \times A(\tau) d\tau, \end{aligned}$$

for every $(t, s) \in [0, T]^2$. In particular, $Z(\cdot, \cdot) : [0, T]^2 \rightarrow \mathbb{R}^{n,n}$ is continuous.

Proof. The first equality is obvious since it corresponds to the definition of a global solution of $(\text{BFCP}_{A,s})$. From this equality and from the classical Gronwall lemma, one can easily prove that $Z(\cdot, \cdot)$ is bounded on $[0, T]^2$. For every $(t, s) \in [0, T]^2$, we introduce

$$T(t, s) := \text{Id}_n + \int_s^t Z(t, \tau) \times A(\tau) d\tau,$$

that is well-defined since $Z(\cdot, \cdot)$ is bounded on $[0, T]^2$. Our aim is to prove that $Z(t, s) = T(t, s)$. From the first equality, it holds that

$$\int_s^t Z(t, \tau) \times A(\tau) d\tau = \int_s^t A(\tau) d\tau + \int_s^t \int_\tau^t A(\xi) \times Z(\xi, \tau) \times A(\tau) d\xi d\tau,$$

for every $(t, s) \in [0, T]^2$. Using the classical Fubini formula (and inverting the roles of τ and ξ), we obtain that

$$\int_s^t Z(t, \tau) \times A(\tau) d\tau = \int_s^t A(\tau) \left[\text{Id}_n + \int_s^\tau Z(\tau, \xi) \times A(\xi) d\xi \right] d\tau.$$

Finally, adding Id_n in the above equality, we obtain that $T(\cdot, \cdot)$ satisfies

$$T(t, s) = \text{Id}_n + \int_s^t A(\tau) \times T(\tau, s) d\tau,$$

for every $(t, s) \in [0, T]^2$. From uniqueness of the global solution of $(\text{BFCP}_{A,s})$, we obtain that $T(t, s) = Z(t, s)$. To conclude, from the definition of $Z(\cdot, \cdot)$, it is clear that $Z(\cdot, \cdot)$ is (absolutely) continuous in its first variable. Using the second equality and the classical Lebesgue dominated convergence theorem, one can easily prove that $Z(\cdot, \cdot)$ is continuous on $[0, T]^2$. \square

Remark 8. From Lemma 16, note that $Z(t, \cdot)$ is the unique global solution of the backward/forward linear Cauchy problem given by

$$\begin{cases} \dot{Z}(s) = -Z(s) \times A(s), & \text{a.e. } s \in [0, T], \\ Z(t) = \text{Id}_n, \end{cases}$$

for every $t \in [0, T]$.

¹⁰This results follows from the classical linear version of the Cauchy-Lipschitz (or Picard-Lindelöf) theorem.

D.2 Recalls on linear Cauchy problems

Let $B \in L^\infty([0, T], \mathbb{R}^n)$ and $q_0, p_T \in \mathbb{R}^n$. From the classical linear version of the Cauchy-Lipschitz (or Picard-Lindelöf) theorem, the forward linear Cauchy problem $(\text{FCP}_{A,B})$ given by

$$\begin{cases} \dot{q}(t) = A(t) \times q(t) + B(t), & \text{a.e. } t \in [0, T], \\ q(0) = q_0, \end{cases} \quad (\text{FCP}_{A,B})$$

admits a unique maximal solution that is moreover global. Similarly, the backward linear Cauchy problem $(\text{BCP}_{A,B})$ given by

$$\begin{cases} -\dot{p}(t) = A(t)^\top \times p(t) + B(t), & \text{a.e. } t \in [0, T], \\ p(T) = p_T, \end{cases} \quad (\text{BCP}_{A,B})$$

also admits a unique maximal solution that is moreover global.

Proposition 8 (Duhamel formulas). *The global solutions of $(\text{FCP}_{A,B})$ and $(\text{BCP}_{A,B})$ are given by*

$$q(t) = Z(t, 0) \times q_0 + \int_0^t Z(t, s) \times B(s) \, ds,$$

and

$$p(t) = Z(T, t)^\top \times p_T + \int_t^T Z(\tau, t)^\top \times B(\tau) \, d\tau,$$

for every $t \in [0, T]$, where $Z(\cdot, \cdot)$ is the state-transition matrix associated to A .

Proof. Let $q : [0, T] \rightarrow \mathbb{R}^n$ be defined by

$$q(t) := Z(t, 0) \times q_0 + \int_0^t Z(t, s) \times B(s) \, ds,$$

for every $t \in [0, T]$. Replacing the value of $Z(\cdot, \cdot)$ by the first equality given in Lemma 16 and using the classical Fubini formula, one can easily prove that

$$q(t) = q_0 + \int_0^t A(\tau) \times q(\tau) + B(\tau) \, d\tau,$$

for every $t \in [0, T]$. As a consequence, q is the unique global solution of $(\text{FCP}_{A,B})$. Similarly, let $p : [0, T] \rightarrow \mathbb{R}^n$ be defined by

$$p(t) := Z(T, t)^\top \times p_T + \int_t^T Z(\tau, t)^\top \times B(\tau) \, d\tau,$$

for every $t \in [0, T]$. Replacing the value of $Z(\cdot, \cdot)$ by the second equality given in Lemma 16 and using the classical Fubini formula, one can easily prove that

$$p(t) = p_T + \int_t^T A(\tau)^\top \times p(\tau) + B(\tau) \, d\tau,$$

for every $t \in [0, T]$. As a consequence, p is the unique global solution of $(\text{BCP}_{A,B})$. \square

D.3 Linear Cauchy-Stieltjes problems

Let $q_0, p_T \in \mathbb{R}^n$. Let $B := (B_i)_{i=1,\dots,j}$ and $\eta := (\eta_i)_{i=1,\dots,j}$ where $B_i \in C_n$ and $\eta_i \in \text{NBV}_1$ is monotonically increasing for every $i = 1, \dots, j$. We say that q is a *global solution* of the forward linear Cauchy-Stieltjes problem $(\text{FCSP}_{A,B})$ given by

$$\begin{cases} dq = A \times q \, dt + \sum_{i=1}^j B_i \, d\eta_i, & \text{on } [0, T], \\ q(0) = q_0, \end{cases} \quad (\text{FCSP}_{A,B})$$

if $q \in \text{BF}_n$ and q satisfies

$$q(t) = q_0 + \int_0^t A(\tau) \times q(\tau) \, d\tau + \sum_{i=1}^j \int_0^t B_i(\tau) \, d\eta_i(\tau),$$

for every $t \in [0, T]$. In such a case, it follows from Lemmas 14 and 15 that $q \in \text{BV}_n$.

Similarly, we say that p is a *global solution* of the backward linear Cauchy-Stieltjes problem $(\text{BCSP}_{A,B})$ given by

$$\begin{cases} -dp = A^\top \times p \, dt + \sum_{i=1}^j B_i \, d\eta_i, & \text{on } [0, T], \\ p(T) = p_T, \end{cases} \quad (\text{BCSP}_{A,B})$$

if $p \in \text{BF}_n$ and p satisfies

$$p(t) = p_T + \int_t^T A(\tau)^\top \times p(\tau) \, d\tau + \sum_{i=1}^j \int_t^T B_i(\tau) \, d\eta_i(\tau),$$

for every $t \in [0, T]$. In such a case, it follows from Lemmas 14 and 15 that $p \in \text{BV}_n$.

Proposition 9. *Problem $(\text{FCSP}_{A,B})$ admits a unique global solution. Problem $(\text{BCSP}_{A,B})$ admits a unique global solution.*

Proof. In this proof, we only treat Problem $(\text{FCSP}_{A,B})$. Let us consider the functional given by

$$\begin{aligned} \mathcal{G} : \text{BF}_n &\longrightarrow \text{BV}_n \subset \text{BF}_n \\ q &\longmapsto \mathcal{G}(q) : [0, T] \longrightarrow \mathbb{R}^n \\ t &\longmapsto q_0 + \int_0^t A(\tau) \times q(\tau) \, d\tau + \sum_{i=1}^j \int_0^t B_i(\tau) \, d\eta_i(\tau). \end{aligned}$$

Note that \mathcal{G} is well-defined from Lemmas 14 and 15. Our aim is to prove that \mathcal{G} admits a unique fixed point. To do so, we will prove that \mathcal{G} admits a contractive iterate. One can easily prove by induction on $k \in \mathbb{N}^*$ that

$$\|\mathcal{G}^k(q_2)(t) - \mathcal{G}^k(q_1)(t)\|_{\mathbb{R}^n} \leq \frac{\|A\|_{L^\infty}^k}{(k-1)!} \int_0^t (t-\tau)^{k-1} \|q_2(\tau) - q_1(\tau)\|_{\mathbb{R}^n} \, d\tau,$$

for every $q_1, q_2 \in \text{BF}_n$, every $t \in [0, T]$ and every $k \in \mathbb{N}^*$. As a consequence, it holds that

$$\|\mathcal{G}^k(q_2) - \mathcal{G}^k(q_1)\|_\infty \leq \frac{(\|A\|_{L^\infty} T)^k}{k!} \|q_2 - q_1\|_\infty,$$

for every $q_1, q_2 \in \text{BF}_n$ and every $k \in \mathbb{N}^*$. Taking $k \in \mathbb{N}^*$ sufficiently large to get $\frac{(\|A\|_{L^\infty} T)^k}{k!} < 1$, we obtain that \mathcal{G}^k is a contractive iterate of \mathcal{G} . Since $(\text{BF}_n, \|\cdot\|_\infty)$ is a Banach space, we conclude that \mathcal{G} admits a unique fixed point. \square

Proposition 10 (Duhamel-type formulas). *The global solutions of (FCSP_{A,B}) and (BCSP_{A,B}) are given by*

$$q(t) = Z(t, 0) \times q_0 + \sum_{i=1}^j \int_0^t Z(t, s) \times B_i(s) d\eta_i(s),$$

and

$$p(t) = Z(T, t)^\top \times p_T + \sum_{i=1}^j \int_t^T Z(\tau, t)^\top \times B_i(\tau) d\eta_i(\tau),$$

for every $t \in [0, T]$, where $Z(\cdot, \cdot)$ is the state-transition matrix associated to A .

Proof. From the Fubini-type formulas provided in Appendix C for Stieltjes integrals, the proof is exactly the same than in Proposition 8. \square

References

- [1] J.E. Ackermann. *Sampled-data control. Volume 1*. Springer-Verlag, Berlin-New-York, 1983.
- [2] G. Bachman and L. Narici. *Functional analysis*. Academic Pr, 1966.
- [3] J.F. Bonnans and C. De La Vega. Optimal control of state constrained integral equations. *Set-Valued Analysis*, 18(3):307–326, 2010.
- [4] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer, New York, 2011.
- [5] F.E. Burk. *A garden of integrals*. Mathematical Association of America, 2007.
- [6] J.B. Conway. *A course in abstract analysis*. American Mathematical Society, 2012.
- [7] J. Faraut. *Calcul intégral*. EDP Sciences, 2006.
- [8] I. Fonseca and G. Leoni. *Modern methods in the calculus of variations: L^p spaces*. Springer, 2007.
- [9] A. Fryszkowski. *Fixed point theory for decomposable sets*. Springer Netherlands, 2004.
- [10] B.V. Limaye. *Functional analysis*. New Age International, 1996.
- [11] X. Li and J. Yong. *Optimal control theory for infinite dimensional systems*. Birkhäuser Boston, 1995.
- [12] B. Mordukhovich. *Variational Analysis and Generalized Differentiation I*. Springer-Verlag, Berlin Heidelberg, 2006.
- [13] J.-P. Penot. *Calculus without derivatives*. Springer-Verlag, New York, 2013.
- [14] W. Sierpinski. Sur les fonctions d'ensemble additives et continues. *Fundamenta Mathematicae*, 3:240–246, 1922.
- [15] R.-L. Wheeden and A. Zygmund. *Measure and integral: an introduction to real analysis*. Monographs and textbooks in pure and applied mathematics, 2015.